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LONDON MATHEMATICAL SOCIETY.

FINANCIAL REPORT FOR THE SESSION 1887-88 (NOV. 10TH, 1887, TO NOV. 8TH, 1888).

CASH ACCOUNT.

Dr.

				£.	s.	d.
Balance from 1886-87:—						
General Fund	£215	9	4			
De Morgan Medal Fund...	1	11	5			
				217	0	9
Interest on Capital—						
Lord Rayleigh's Fund ...	54	14	2			
Life Compositions Fund...	34	14	2			
Invested Surplus Fund ...	14	10	1			
De Morgan Medal Fund ...	3	7	11			
				107	6	4
9 Entrance Fees				9	9	0
75 Subscriptions—						
1 for 1883–84	£1	1	0			
4 for 1884–85	4	4	0			
9 for 1885–86	9	9	0			
28 for 1886–87	29	8	0			
29 for 1887–88	30	9	0			
4 for 1888–89	4	4	0			
				78	15	0
Sales of *Proceedings*				106	8	2
8 Life Compositions				84	0	0
				£602	19	3

Cr.

				£.	s.	d.
Printing *Proceedings*, &c.				215	3	6
Purchase of Journals				7	18	6
Binding				5	18	6
Postage, Stationery, and Sundries				20	13	11
Erecting Shelves				3	9	0
Legal Expenses of Petition for Charter ...				48	17	6
Allowance to Member for *Proceedings*				2	14	0
Rent				10	0	0
Coals and Gas				1	10	0
Attendance—						
British Association ...	£1	5	0			
Mr. Stewardson	10	10	0			
				11	15	0
Life Compositions Fund—						
Purchase of £83. 10s. 4d. Consols ...				84	0	0
Invested Surplus Fund—						
Purchase of £146. 3s. 3d. Consols ...				150	0	0
Balance at Bank—						
General Fund	£36	0	0			
De Morgan Medal Fund	4	19	4			
				40	19	4
				£602	19	3

Audited and found correct,

21*st November*, 1888. (Signed) GEORGE HEPPEL.

General Fund.

				£.	s.	d.					£.	s.	d.
Cash at Bank				36	0	0	4 Subscriptions for 1888-89 paid in advance				4	4	0
129 Subscriptions owing—							27 Subscriptions struck off as irrecoverable—						
3 for 1883–4	£3	3	0				3 for 1883–4	£3	3	0			
9 for 1884–5	9	9	0				6 for 1884–5	6	6	0			
13 for 1885–6	13	13	0				6 for 1885–6	6	6	0			
21 for 1886–7	22	1	0				6 for 1886–7	6	6	0			
*83 for 1887–8	87	3	0				6 for 1887–8	6	6	0			
				135	9	0					28	7	0
1 Entrance Fee owing				1	1	0	Balance				139	19	0
				£172	10	0					£172	10	0

* Viz., 116, the number of Subscribers for 1887-88, less 4 Subscriptions paid in advance in 1886-87, and 29 paid in 1887-88.

De Morgan Medal Fund.

	£.	s.	d.		£.	s.	d.
Cash at Bank	4	19	4	Balance	4	19	4

CAPITAL ACCOUNT.

	Sum Invested.						Description of Investment.
	£.	s.	d.	£.	s.	d.	
General Fund—							
1. Lord Rayleigh's Fund	1000	0	0	870	0	0	Guaranteed Five per Cent. Great Indian Peninsula Railway Stock.
2. Life Compositions Fund	931	0	0	949	2	7	2¾ per Cent. Consols, Account A.
3. Invested Surplus Fund	500	0	0	496	19	6	" " " " B.
De Morgan Medal Fund	103	5	3	104	19	8	" " " " C.

Audited and found correct,

21*st November*, 1888. (Signed) GEORGE HEPPEL.

PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XX.

FROM NOVEMBER, 1888, TO NOVEMBER, 1889.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XX.

TWENTY-FIFTH SESSION, 1888—1889.

November 8th, 1888.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street, W.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

The Treasurer read his Report. Its reception was moved by S. Roberts, F.R.S., seconded by Prof. Lloyd Tanner, and carried unanimously.

At the request of the Chairman, Mr. Heppel consented to act as Auditor.

From the Report of the Secretaries, it appeared that the number of the members on the List, after striking off the names of defaulters, was 190, of these 80 being life-members.

The Society had lost two members by death—Mr. J. Brooksmith, M.A., of Cheltenham College, on May 5th; and Mr. Arthur Buchheim, M.A., a member of Council, on September 9th.

The following communications had been made:—

On Pure Ternary Reciprocants, and Functions allied to them: E. B. Elliott, M.A.

On the General Linear Differential Equation of the Second Order: the President.

On the Stability of a Liquid Ellipsoid which is rotating about a Principal Axis under the Influence of its own Attraction: A. B. Basset, M.A.

(1) On $\kappa\lambda-\kappa'\lambda'$ Modular Equations, (2) Geometry of the Quartic: R. Russell, M.A.

The Differential Equations satisfied by Concomitants of Quantics: A. R. Forsyth, F.R.S.

On the Stability or Instability of certain Fluid Motions: Lord Rayleigh, Sec. R. S.

Notes on a System of Three Conics touching at One Point: Dr. J. Wolstenholme.

The Algebra of Linear Partial Differential Operators: Captain P. A. Macmahon, R.A.

On a Method in the Analysis of Ternary Forms: J. J. Walker, F.R.S.

Confocal Paraboloids: Prof. Greenhill, M.A.

Harmonic Decomposition of Functions, and some Allied Expansions: A. R. Johnson, M.A.

Uni-Brocardal Triangles, and their Inscribed Triangles: R. Tucker, M.A.

The Theory of Distributions: Captain P. A. Macmahon, R.A.

On the Analogues of the Nine-points Circle in Space of Three Dimensions, and connected Theorems: S. Roberts, F.R.S.

On a Theorem analogous to Gauss's in Continued Fractions, with applications to Elliptic Functions: L. J. Rogers, M.A.

A Theorem connecting the Divisors of a certain Series of Numbers: Dr. Glaisher, F.R.S.

On Reciprocal Theorems in Dynamics: Prof. H. Lamb, F.R.S.

The Free and Forced Vibrations of an Elastic Spherical Shell containing a given Mass of Liquid: A. E. H. Love, B.A.

On the Volume generated by a Congruency of Lines: R. A. Roberts, M.A.

Isoscelians: R. Tucker, M.A.

Complex Multiplication Moduli of Elliptic Functions: Prof. Greenhill, M.A.

A Case of Complex Multiplication with Imaginary Modulus arising out of the Cubic Transformation in Elliptic Functions: Prof. Cayley, F.R.S.

Geometrical Proof of Feuerbach's Theorem concerning the Nine-point Circle: Prof. Genese, M.A.

A Group of Isostereans: R. Tucker, M.A.

On Simplicissima in Space of n-Dimensions: W. J. C. Sharp, M.A.

Synthetical Solutions in the Conduction of Heat: E. W. Hobson, M.A.

On certain Operators in connection with Symmetric Functions (Supplementary Note): R. Lachlan, M.A.

On a Law of Attraction which might include both Gravitation and Cohesion: G. S. Carr, M.A.

Some Theorems on Parallel Straight Lines, together with some attempts to prove Euclid's Twelfth Axiom: J. Cook Wilson, M.A.

On Cyclicants, or Ternary Reciprocants, and allied Functions: E. B. Elliott, M.A.

On the Flexure and the Vibrations of a Curved Bar: Prof. H. Lamb, F.R.S.

On the Figures formed by the Intercepts of a System of Straight Lines in a Plane, and on analogous relations in Space of Three Dimensions: S. Roberts, F.R.S.

(1) Lamé's Differential Equation, and (2) Stability of Orbits: Prof. Greenhill, M.A.

On the Determination of the Circular Points at Infinity: Rev. Dr. C. Taylor.

On the c- and p-Discriminants of Integrable Differential Equations of the First Order: Prof. M. J. M. Hill, M.A.

On Point-, Line-, and Plane-Sources of Sound: Lord Rayleigh, Sec. R. S.

Note on Rationalisation: H. Fortey, M.A.

Applications of Elliptic Functions to the Theory of Twisted Quartics: Prof. G. B. Mathews, M.A.

Coefficients of Induction and Capacity, and allied Problems: Prof. Greenhill, F.R.S.

Electrical Oscillations: Prof. J. J. Thomson, F.R.S.

Demonstration of the Theorem that the Equation $x^3+y^3+z^3=0$ cannot be solved in Integers: J. R. Holt, B.A.

Additional exchanges of *Proceedings* were made with the Circolo Matematico of Palermo, and the Faculté des Sciences de Toulouse.

The same Mathematical Journals had been subscribed for as in the preceding Session.

The meeting next proceeded to the election of the new Council.

The Scrutators (Rev. J. J. Milne and Mr. S. O. Roberts) having examined the Balloting Lists, declared the following gentlemen duly elected:—President, J. J. Walker, F.R.S.; Vice-Presidents, Sir James Cockle, Knt., F.R.S., E. B. Elliott, M.A., A. G. Greenhill, F.R.S.; Treasurer, A. B. Kempe, F.R.S.; Secretaries, M. Jenkins, M.A., R. Tucker, M.A.; other Members of the Council, A. B. Basset, M.A., J. W. L. Glaisher, Sc.D., F.R.S., J. Hammond, M.A., Prof. Hart, M.A., Joseph Larmor, D.Sc., C. Leudesdorf, M.A., Captain P. A. Macmahon, R.A., Samuel Roberts, F.R.S., E. Routh, Sc.D., F.R.S.

Mr. Walker, on taking the Chair, thanked the Society for the honour they had conferred upon him, and then called upon Sir James Cockle to read his Presidential Address, "On the Confluences and Bifurcations of certain Theories." On the motion of Dr. Glaisher, seconded by Mr. S. Roberts, it was carried by acclamation, that the Address, with the Author's permission, be printed in the *Proceedings*.

The following communications were made:—

On Cyclotomic Functions: § 1. Groups of Totitives of n, § 2. Periods of n^{th} Roots of Unity: Prof. Lloyd Tanner.

On a Theory of Rational Symmetric Functions: Captain P. A. Macmahon, R.A.

The Factors and Summation of $1^r+2^r+\ldots+n^r$: The Rev. J. J. Milne.

Raabe's Bernoullians: Mr. J. D. H. Dickson.

Certain Algebraical Results deduced from the Geometry of the Quadrangle and Tetrahedron: Dr. Wolstenholme.

On a certain Atomic Hypothesis: Prof. K. Pearson.

On Deep-water Waves resulting from a Limited Original Disturbance: Prof. W. Burnside.

The following presents were received:—

Cabinet likeness of Captain Macmahon.

"Royal Society, Proceedings," No. 271.

"Physical Society of London, Proceedings," Vol. IX., Part IV.; October, 1888.

"Manchester Literary and Philosophical Society, Memoirs and Proceedings," Fourth Series, Vol. I.; Manchester, 1888.

"Annales de la Faculté des Sciences de Toulouse," Tome II., Fasc. 3 and 4; Paris, 1888.

"Bulletin des Sciences Mathématiques," Tome XII.; July, 1888.

"Beiblätter zu den Annalen der Physik und Chemie," Band XII., St. 9 and 10.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. IV., Fasc. 11 and 12.

"Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," XXI.—XXXVII.

"Memorie della Regia Accademia di Scienze, Lettere, ed Arti in Modena," Serie II., Vol. V.; Modena, 1887.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VIII., No. 5; Coimbra, 1887.

"Memorias de la Sociedad Cientifica – Antonio Alzate," Tomo II., Nos. 2 and 3; Mexico, 1888.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Nos. 67 and 68.

"Rendiconti del Circolo Matematico di Palermo," Tomo II., Fasc. 5; Settembre, Ottobre.

"Teoría de los Errores," par H. Faye, 8vo pamphlet; Mexico, 1888.

Pamphlets by Em. Lemoine:—"De la Mesure de la Simplicité dans les Sciences Mathématiques;" "Notes sur diverses Questions de la Géométrie du Triangle," (Association Française, 1888.)

"Note sur l'usage des Coordonnées dans l'Antiquité et sur l'Invention de cet Instrument," par H. G. Zeuthen, 8vo pamphlet; Copenhagen, 1888.

"On the Diameters of a Plane Cubic," by J. J. Walker, F.R.S. ("Phil. Trans.," Vol. CLXXIX., A. pp. 151—203).

"Lectures on the Ikosahedron, and The Solution of Equations of the 5th Degree," by Felix Klein; from the Translator.

"Educational Times," November, 1888.

On the Confluences and Bifurcations of certain Theories. (*Presidential Address.*) *By* Sir JAMES COCKLE, F.R.S.

[*Read November 8th*, 1888.]

Axioms, says Proclus,[1] are common to all sciences, though each employs them in its peculiar subject-matter. A little further on[2] he cites Aristotle[3] as saying that one science is more certain than another, viz., that which emanates from more simple suppositions than that

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[1] Proclus, "Commentaries on the First Book of Euclid's Elements" (Taylor's Translation, London, 1792), p. 92.

[2] Proclus, *op. cit.*, p. 93.

[3] Taylor (*ib.* p. 93) supplies the reference to the first Analytics, t. 42.

which uses more various principles; and that which tells the why, than that which tells only the simple existence of a thing; and that which is conversant about intelligibles, than that which touches and is employed about sensibles.

Proclus adds that, according to these definitions of certainty, arithmetic is more certain than geometry, since its principles excel by their simplicity. For the conception of unity has no reference to position in space, while that of a point involves such reference. In short, we may say that to count a number of objects is a simpler operation than to measure the distances between them.

All this, and much more, shows how early the notion of what is sometimes called a hierarchy of the sciences arose. Proclus's order of precedence would seem to be this, viz., logic,[1] arithmetic, geometry, mechanics, optics, dioptrics,[2] and so on; the progression being from the more to the less abstract, or from the abstract to the concrete.

Francis Bacon, mindful perhaps of Proclus,[3] and duly appreciating the power of mathematics as an instrument[4] and its value as a discipline,[5] expressly takes the degree of abstractness of a science as the mark for its classification. He places mathematics, as the most abstract of sciences,[6] at one end of the scale and "policy" at the other. He does not graduate the scale minutely, but it may be that, as in the case of the categories,[7] he attached no great value to such details. Distinguishing philosophy from theology, logic, and mathematics,[8] he assigns to it the axioms which are common to several sciences and the inquiry into essences, as quantity, similitude, diversity, possibility, and the rest. Science he divides between metaphysics, the science of the abstract and permanent, and physics, that of matter and its changes.[9] Bacon, in one place, names the one universal science by the name of philosophy, while in another he treats philosophy and metaphysics as two distinct things.[10] He uses the word metaphysics in a sense different from that in which it was then[11] received. Mathematics he places as a branch of metaphysics,

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[1] Proclus, *op. cit.*, p. 79. Hume (Treatise, Vol. I., Lond. 1739, Bk. I., Pt. III., p. 129, *et vid.* p. 128) says that geometry falls short of that perfect precision and certainty which are peculiar to arithmetic and algebra.

[2] Proclus, *op. cit.*, p. 93; *et vid.* pp. 78, 79.

[3] Bacon, "The Proficience and Advancement of Learning" (Oxford, 1633,) pp. 49, 50.

[4] Bacon, *op. cit.*, pp. 151, 152; *et vid.* pp. 119, 120.

[5] Bacon, *op. cit.*, pp. 152, 205, and 231.

[6] Bacon, *op. cit.*, p. 218; *et vid.* pp. 150, 151.

[7] Bacon, *op. cit.*, pp. 130, 131, 140, and 201; *et vid.* p. 161.

[8] Bacon, *op. cit.*, pp. 49, 50; *et vid.* pp. 130, 131, and 140.

[9] Bacon, *op. cit.*, p. 141.

[10] Bacon, *op. cit.*, pp. 130, 140.

[11] Bacon, *op. cit.*, p. 138; *conf.* pp. 146, 147.

and as having determined or determinate quantity for its subject. To the pure mathematics, he says, belong geometry and arithmetic; the one handling continuous, and the other discrete quantity.[1] If he means continuous quantity so far as it is immovable, he agrees with the Pythagoreans.[2]

Quantity, time, and space are placed by Aristotle among his categories, or are implied in them. With regard to space, he does not seem to have reached the Kantian view in any way, nor to be very clear in his meaning, though he apparently feels that to realise space we must have motion. His conception of time as one of the elements required for measuring motion, and his starting the problem as to whether we could have time without a mind to conceive, seem a more distinct approximation, though only an approximation, to Kant's view of time as merely a subjective condition of perception.[3]

Newton, in the Scholium to his Definitions, distinguishes between absolute and relative time, the latter being time conceived in its relation to phenomena. Of absolute time (otherwise called duration) which has no relation to anything external, he says that it flows equably, and that its rate of flow and the order of its parts are immutable. In his "Fluxions" he uses the word time in a somewhat different sense, viz., as meaning the independent variable, characterised by an equable increase, fluxion, or flow.[4] Sir W. Rowan Hamilton treated algebra as the Science of Pure Time, but his doctrine is not entirely[5] assented to by De Morgan, nor by Professor Cayley, who indeed, in his Southport Address (page 19), intimates dissent from it. Proclus does not connect arithmetic with time, and Professor Cayley suggests (*ib.* p. 18) that, in any case, the notion of number or plurality is not more dependent on time than on space. By the

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[1] Bacon, *op. cit.*, pp. 150, 151.

[2] Proclus (Taylor's Translation), p. 74.

[3] For this summary of Aristotle's views I am indebted to Mr. Reginald H. Roe, who referred me to Ueberweg's "Hist. of Phil.," p. 164, for a more general statement, and to p. 165 for a list of the best books for its fuller elucidation, adding that in Ritte and Preller's extracts, pp. 288 and 289, will be found all the important passages from Aristotle bearing on the question. As to the views of Boole, see his "Laws of Thought" (London, 1854), pp. 162 *et seq.*; see also p. 419. Boole treats of space at pp. 163, 175, and 418; and at p. 175 he quotes Aristotle's statements respecting the existence of space in three dimensions.

[4] Newton, "Fluxions," pp. 26 and 38 of the small edition (London, 1737). This is a genuine work of Newton's. As to its bibliography, see *Notes and Queries*, 2nd S., Vol. x., pp. 163, 232, 233; 3rd S., Vol. xi., pp. 514, 515; 4th S., Vol. ii., p. 316; 5th S., Vol. iv., p. 401; 6th S., Vol. iv., pp. 129, 130; Vol. v., pp. 263, 264, 304, 305, and 426. This octavo edition is very scarce. Indeed, I only know of two copies, viz., my own copy and one in the library of the Royal Astronomical Society.

[5] De Morgan, "On the Foundation of Algebra," *Cambridge Transactions*, Vol. vii., pp. 173–187; see pp. 175, 176. The remarks of Prof. Cayley on Whewell, at p. 18 of his Southport Address, are applicable to Rowan Hamilton.

logicians, time seems to be regarded as the more abstract of the pure intuitions. In fact, time is implied in memory and in thought itself, and Professor Francis W. Newman observes that no man could get through a syllogism if he forgot the first premiss while dwelling on the second.[1] Moreover, he has recourse to the idea of time when he comes to discuss propositions,[2] and Boole investigates the nature of the connection of his own Secondary Propositions with the idea of time.[3] The ancient Indians had their cyclical periods, but not therefore necessarily any notion of a uniform curvature (so to say) of time.

Absolute space, says Newton, perpetually remains similar to itself and immovable; and, further on in the Scholium, he adds that the order of its parts is immutable. In the Preface to the Principia he had observed that the description of straight lines and circles, on which geometry is founded, belongs to mechanics, and he follows up this train of thought. But, whether he means to detach himself from Plato, I must leave others to say. It is said to be certain that he was familiar with Bacon's works; that he uses the word axiom, not in Euclid's sense, but in Bacon's, thus giving the name of axioms to the laws of motion, which, of course, are ascertained by the scrutiny of nature, and to those general experimental truths which form the groundwork of optics.[4] Now Bacon says that, in his judgment, the senses are sufficient to certify and report truth, either immediately or by way of comparison.[5] Moreover, he suggests that the rule *Quæ in eodem tertio conveniunt, et inter se conveniunt*, a rule so potent in logic as that all syllogisms are built upon it, is taken from the mathematics.[6] In seeking an origin for the more abstract in the less abstract, Bacon is not solitary. Thomas Stephens Davies suggested[7] that the argument from superposition had its origin in mechanical considerations, and from the fitting together of material figures. Moreover, it is conceivable that some observant person among the ancient Egyptians, whose custom it was to stamp their bricks, noticing the resemblances of the marks and the correspondence of the impressions with the

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[1] Newman, "Lectures on Logic, or on the Science of Evidence," etc. (Oxford, 1838), p. 15.

[2] Newman, *op. cit.*, pp. 32—34.

[3] Boole, "An Investigation of the Laws of Thought" (London, 1854), pp. 162, *et seq.*

[4] See the Account of the "Novum Organon" in the Library of Useful Knowledge, p. 10.

[5] Bacon, "Advancement of Learning" (cited *supra*), p. 193.

[6] Bacon, *op. cit.*, p. 132.

[7] T. S. Davies, Geometrical Notes, "Mechanics' Magazine," Vol. LIII. (1850), pp. 150, 169, 262, 291, 442. Davies points out "the connection between parallels and similar triangles." He thinks that Aristotle's secession from the school of Plato arose from his enforcement of his own logical doctrines. Davies rejects the notion of a geometry built upon definitions alone without the assistance of axioms.

impressing tool, may have been led to a recognition of the rule quoted by Bacon. The doctrine that there enters into geometry an element derived from the senses has, indeed, appeared in books designed for ordinary readers. Thus, Prof. Newman, writing in 1836–38, although in one part of his "Logic" (p. 25), he says that in geometry no results are admitted by help of observation and testimony, but only by reasoning from the definition, yet he afterwards (p. 55) states that, as space and its properties appear undeniably to be learned by sense, the argument seems to him to preponderate for naming geometry a Mixed Science, and believing that its propositions are real and not verbal truths. And Potts[1] says that geometry seems to rest on the simplest inductions from experience and observation, and that its principles are founded on facts cognisable by the senses.

But it is to Reid[2] that the idea of a more precise mathematical treatment of the subject is due, and his name ought to head the roll on which will be inscribed the names of Lobatschewsky, Riemann, and other investigators. Kant, indeed, disposes of such questions summarily, by saying that it follows from his premisses that the propositions of geometry are not the determinations of a mere creature of our feigning fancy, but that they necessarily hold of space, and consequently of all that may be met within it, because space is nothing else than the form of all the external phenomena, in which alone objects of sense can be given ("Prolegomena,"[3] p. 51). He adds (pp. 51 and 53) that external phenomena must necessarily and precisely agree with the propositions of the geometer. Whether Kant's allusion to "superficial metaphysicians" points to the Pyrrhonists and Epicureans[4] or to others, and, possibly, even to Reid, whom he had mentioned before (Pref., p. viii.), does not appear. Whatever opinions be formed of Kant's theory, or of the nature of space, his view is impressive. Confine that view to two dimensions, and suppose the surface of a sphere to be inhabited by a being destitute of any conception of a third dimension, and whose senses are unaffected by any point not situated or any motion not taking place on that

[1] Potts (Robert), "Euclid's Elements of Geometry," etc. (Cambridge and London, 1845); Notes to Book I., p. 41.

[2] Thomas Reid, "An Inquiry into the Human Mind on the Principles of Common Sense" (1764). My pagings refer to the Calcutta Reprint of 1869. Chapter VI. treats (pp. 94—277) of Seeing; its Sect. VII. (pp. 120—124), of Visible Figure and Extension; and its Section IX. (pp. 132—145), of the Geometry of Visibles. In Sect. VIII. (pp. 125—132), we have Some Queries concerning Visible Figure answered.

[3] I cite from Richardson's Translation (London, 1819); and cannot now give the corresponding paging in that of Prof. Mahaffy.

[4] Montucla, "Histoire" (2^nde édition, An. VII.), p. 21.

surface. He could only estimate direction and position by the tangent to the path of the visual ray at the point where that path meets his visual organ, and would think that all objects were situate in one plane. His geometry would be Euclidian; for, if he could form a notion of the actual paths of rays, he would have a conception of the third dimension in space.[1] Here Kant and Riemann would apparently be at issue; for, if a more general conception of space is to be rendered special by actual measurements on the sphere, then, after an enlarged experience, the Euclidian conception would have to be expelled and replaced by some other. And all this would have to be done without praying in aid the excluded third dimension.

Aristotle[2] notices that the nature of everything is best seen in its smallest portions, and Kant[3] remarks that there was a time when mathematicians, who were philosophers too, began to doubt, not the truth of their geometrical propositions as far as they regard space, but the objective validity and applicability of the conception itself, and of all its determinations, to nature; as they were apprehensive that a line in nature might consist of physical points, and that consequently true space in the object might consist of simple parts, though space as conceived by the geometer cannot so consist. Clifford[4] would have given due weight to the doubts of the philosophical mathematicians. He even suggests that the properties of space may change with time. Now, a number may be a function of an angle; the very angle itself determines those numbers (ratios of lines) which we call sines and cosines. But, says De Morgan,[5] in every case but this it is impossible to conceive number a function of magnitude. It seems almost equally difficult to entertain Clifford's conjecture, which, nevertheless, measurements might verify. The sentence, *Nam tempora et spatia sunt sui ipsorum et rerum omnium quasi Loca*, in Newton's Scholium, though it may suggest that omnipresence does not involve extension in space, implies no functional relation between space and time. The words "then and there," accompanying every material allegation in indictments, would suffice to show that the opinions of the world at large on certain characteristics

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[1] See Cayley, Southport Address, pp. 11, 12.

[2] See Bacon, "Advancement of Learning," p. 108.

[3] Kant, "Prolegomena," p. 52.

[4] William Kingdon Clifford, "Mathematical Papers" (London, 1882). See pp. xl. and xlii. of the Introduction, by H. J. S. Smith.

[5] De Morgan, "On the General Principles of which the Composition or Aggregation of Forces is a Consequence" (*Camb. Trans.*, Vol. x., Part II., pp. 294, 295, foot-note).

of time and space[1] were in accord with that of the philosophers. Indeed, their isolation, as forms of intuition, may no more be a peculiarity of Kant's system than is his distinction between analytical and synthetical judgments. This distinction was present to the mind of Bacon,[2] as well as to that of Locke, whom Kant cites ("Prolegomena," p. 25), and who, elsewhere than in the place cited, adverts to the distinction. That which Locke had styled a trifling proposition, Kant called an analytical judgment; and that which Locke (Essay concerning Human Understanding, Book IV., Chap. VIII., Sect. 8) styled a real truth, Kant would have called a synthetical judgment. With Hume, too, Kant is in some respects in close relation. Hume (Treatise, Vol. I., Bk. I, Pt. II., pp. 53—124) treats specially of the ideas of space and time. Hume, again (Inquiry, p. 17; Essay IV., p. 50), distinguishes between results attained by reasonings *à priori* and results arising entirely from experience (Inq., p. 17; Ess., p. 49). He seems to allow conception a sufficiently wide range, for he urges (Inq., p. 13; Ess. II., pp. 26, 27) that, in one exceptional instance, there may be an idea not arising from a corresponding impression; viz., in the case when from the impressions of two distinct shades of a particular colour, a conception is formed of an intermediate shade of the same colour. He asserts (*ib.*, p. 118) that the only objects of the abstract sciences or of demonstration are quantity and number.

If, as Clifford[3] seems to think, there are no sufficient grounds for maintaining that, if our space has curvature, it must be contained in a space of more dimensions and no curvature, one difficulty is apparently removed. The one-dimensioned time is something very different to space, from which the higher-dimensioned entity might differ still more; and if a solid be treated as the shadow or projection in Euclid's space of, say, a four-dimensioned body, that part of the body which lies outside the shadow seems to have no quality analogous to impenetrability or inertia, nor indeed any quality which affects the senses or deranges the results of calculation. Prof. Cayley says (Southport Address, p. 11) that Riemann's idea seems to be that of modifying the notion of distance, not that of treating it as a locus in four-dimensional space. The suggestion (Cayley, *ib.* p. 10) of a rule

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[1] I should have been glad to have given Locke's and Kant's descriptions of space and time, and to have compared them with Newton's. But I cannot omit to refer to a Smith's Prize paper, by Mr. Robert Franklin Muirhead, printed in the "Philosophical Magazine" for June, 1887; S. 5, Vol. XXIII., pp. 473—489.

[2] Bacon, "Advancement of Learning," p. 47.

[3] Clifford, "The Universal Statements of Arithmetic" *Nineteenth Century* (1879, Vol. V., pp. 513—522; *vide* p. 522).

changing its length by an alteration of temperature facilitates apprehension. Prof. von Helmholtz has considered the effect of the changes in sensible phenomena which a transition to a spherical or pseudo-spherical world, if such things be, would produce; and he has taken an independent view of the subject in other respects.[1]

De Morgan[2] professed to have been puzzled to know on which side the meeting of parallels took place, or whether on both. He concludes that they never meet. This, however, does not shake, nor is it to be supposed that he wished[3] it to shake, the belief in modern methods, for he apparently admits that interpretation of forms may demand conclusions which can be reached by reasoning on infinity, if increase without limit show approach. He observes that it is clearly conceived by the logicians that all division is reducible to simple dichotomy and its repetitions, and that when the logician has once shown division, difference, he does not trouble himself with the difficulty of repetitions. De Morgan's remark is easily verified by turning to Potts' Note on Euc. I. 10 (p. 49). Turning again to Boole, (L. of T., p. 91), it would seem that the logician does not completely detach himself from the notion of infinity; he has to interpret 1 : 0 as well as 0 : 0.[4]

Bacon differs from Plato, who considered forms as absolutely abstracted from matter, and not as confined and determined by it, and agrees with Aristotle in saying that words are the images of thoughts;[5] so that the agreement of the views of Bacon with those of Prof. Max Müller, would seem to be tolerably close. It is easy to find cases in which a doubtful meaning of a word may give rise to disagreement on matters of substance. Boole (L. of T., pp. 407, 408)

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[1] A paper in *Mind*, by Prof. von Helmholtz, elicited a criticism from Prof. Land, which produced a reply, and with a brief note appended to a paper on another subject, by Prof. Land, the discussion closed. See *Mind*, Vol. I., pp. 301—321; Vol. II., pp. 38—46; Vol. III., pp. 212—225; and Vol. IV.

[2] De Morgan, "On Infinity," etc. (*Camb. Trans.*, Vol. XI., Part I., 1865, pp. 145—189; *vid.* pp. 173, 176, 180, 147). In connection with this paper, the comments of Mr. W. S. B. Woolhouse in the *Educational Times* (*Reprint*, Vol. VI., pp. 49—52) should be considered. And in connection with a paragraph at pp. 161, 162 of De Morgan's paper, the leading paragraph of p. 424 of a previous paper of his, "On the Theory of Errors of Observation" (C. T., Vol. X., Pt. II., 1862), should be read. In the last-mentioned passage he distinguishes between the zero and the indivisible of probability. Hamilton, of Edinburgh, following earlier authorities, expressly restricts the application of logic to finite things. But it does not therefore follow that logicians in general turn a deaf ear to all reasoning upon infinites and infinitesimals, and that they reject results stamped with authority and universally accepted.

[3] This sufficiently appears from a statement at p. 15 of his paper, "On the Root," &c. (*Camb. Trans.*, Vol. XI., Pt. II.).

[4] See the last footnote but one.

[5] Bacon, "Advancement of Learning," p. 143; *Conf.*, pp. 130, 140. See also pp. 192, 209.

observes that the term "necessary" may be applied either to the observed constancy of nature or to the logical connection of propositions. He expresses no decided preference for either meaning. The meanings should be kept carefully apart. If an axiom be a necessary truth, in the strictest sense, then Newton's laws of motion are laws *à priori*, viz., giving Kant's meaning to the term (Prol., p. 103), they are known independently of all experience. But Laplace (Méc. Cél., pp. 14—18[1]) treats them as results of experience. Moreover, he treats (pp. 65—69) the laws of motion under all the relations mathematically possible between force and velocity. Newton, in fact, usually speaks of "law," and gives the term "axiom," Bacon's meaning.

Boole's Chapter XX. (L. of T., pp. 320—375) relates to problems on causes, but his use of the word "cause" has given rise to much discussion. He proposed a question on causes in 1851, which was answered by Prof. Cayley in 1853. The solution was criticised by Boole in 1854, who arrived at a different result, and, in 1854, Mr. H. Wilbraham examined both solutions. Prof. Cayley returned to the subject in 1862, and Boole thereupon admitted that it would have been better, in stating his problem, not to have employed the word "cause" at all.[2] One mode of stating the nature of the relation between "cause" and "effect," may be this, viz., when a certain (antecedent) change is immediately and invariably followed by a certain other (subsequent) change, then the relation in which the antecedent stands to the subsequent (which may now be called the consequent) change is that of cause and effect. This is, in substance, if not in form, a view common to Algazel, Glanvil,[3] Hume,[4] Brown,[5] Kant, and, as I

[1] My pagings refer to the 2nd ed. of the "Mécanique Céleste," Vol. I. (Paris, 1829).

[2] Boole, "C. and D. M. J.," Vol. VI., p. 286; "L. of T.," pp. 321—326; "Phil. Mag.," S. 4, Vol. VII., pp. 29—32; Vol. XXIII., pp. 361—363; Wilbraham, "Phil. Mag.," S. 4, Vol. VII., pp. 465—476; Cayley, "Phil. Mag.," S. 4, Vol. VI., p. 259; S. 4, Vol. XXIII., pp. 352—365, and 470. A short letter by Boole ("Phil. Mag.," S. 4, Vol. XXIV. (1862), p. 80) concludes the discussion.

[3] Glanvill (Joseph) "Scepsis Scientifica," etc. (Lond. 1665, 4°); Lond. 1885, 8°. On Causation I have only mentioned comparatively recent authors. But, going further back, we find Thales (with his elemental analysis), Xenophanes (with his one cosmic substance), and Pythagoras (with his arithmetical and geometrical combinations), all recognizing invariable sequences in nature; and Socrates admitted a class of phenomena wherein the connection of antecedent and consequent was invariable and ascertainable by human study (Grote, "History of Greece," Vol. I., 1846, pp. 495—498). Socrates applied similar scientific reasonings to moral and social phenomena (*ib.* p. 504).

[4] David Hume, "A Treatise of Human Nature," etc. (Lond., Vols. I. and II., 1739; Vol. III., 1740. His name does not appear on the title-pages). Philosophical Essays concerning Human Understanding" (2nd ed., Lond. 1750). "An Inquiry concerning Human Understanding" (Lond. 1861 marks the issue to which I refer).

[5] Thomas Brown, "Inquiry into the Relation of Cause and Effect" (Third Edi-

believe, Reid; for the question seems to be one about words. It differs but slightly from the view (C. T., Vol. x., Pt. II., p. 300) of De Morgan. Perhaps "unvarying" might be a better word than "invariable," for one instant of time is the immediate and invariable antecedent of its consecutive instant; but the idea of "cause" does not seem to arise. When "cause" is used in the above sense, the solutions of Boole and Prof. Cayley agree. Boole's question has been dealt with in our *Proceedings* (Vol. XI., p. 118) by Mr. McColl.

The import of the word "principle," is not the same when we speak of the Principle of Contradiction or of Excluded Middle, as when we speak of the Principle of the Permanence of Equivalent Forms, or of the Sufficient Reason, or of Continuity. That of sufficient reason has been assailed by Brown (C. and E., Sect. IV., pp. 222, misnumbered 322, to 306), and by De Morgan (C. T., x., Pt. II., pp. 290—304). Clifford (*Op. cit.*, p. xli.) was prepared to sacrifice the principle of continuity, even in the case of space, and the author of anonymous "Strictures" on Peacock's Algebra (Camb., 1837), who was (so at least I was told many years ago by Davies) Hind, concludes (p. 21) that number is perfectly abstract, that it is the only thing which is so, and that it is not rightly denominated a species of quantity, being equally connected with every species. An instance of a striking failure of the principle of the permanence of equivalent forms is given by Dr. J. W. L. Glaisher in the *Messenger of Mathematics*, N. S., Vol II. (1872), p. 95. Again, take another word, viz., "disparity." Supposing it to be said that there are two persons in a room, whose united ages are twenty-one years, and between whose ages there is the greatest disparity possible. This is intelligible if one be a newborn or nascent infant, and the other a person aged twenty-one. But suppose the same statement made of three persons; the proficient in language might have to inquire of the mathematician what meaning, if any, the statement bears. Or, again, the mathematician might be asked what, or whether any, numerically definite meaning can be attached to the words, "triangle of maximum scalenity."

Prof. Newman ("Logic," 1838, p. 52) says that the truths of arithmetic are verbal. Perhaps this, and the corresponding statements of Dugald Stewart, would not now be insisted on. They are opposed to the views of Kant, Clifford, and De Morgan (C. T., XI., Pt. I., p. 160). The identities $3^2+4^2=5^2$ and $3^3+4^3+5^3=6^3$

tion, Edinburgh, 1818). Draper does not admit the construction put upon Algazel's words by Whewell ("Hist. Ind. Sc.," Lond., 1837, I., p. 251). A facsimile reprint of Glanvil has been published within the last few years. Buckle pronounced Brown's to be one of the best books ever written.

seem to be something very different from definitions of words. Kant considers $7+5=12$ to be a synthetical judgment (Proleg., pp. 22, 23).

Metaphysics and mathematics are consorts in the East as well as the West. Bhascara says that the analytical art is merely sagacity exercised, and is independent of symbols, which do not constitute the art.[1] If De Morgan[2] be right in placing Diophantus as late as the beginning of the seventh century, Aryabhatta was earlier, by two centuries, than Diophantus. The name, certainly, seems to have been a very common one. Josephus[3] relates that Alexander (a son of Herod the Great) said that Diophantus the scribe had imitated his hand. But Mr. Heath's work[4] renders it scarcely possible to sustain De Morgan's contention.

On Raabe's Bernoullians. *By* J. D. Hamilton Dickson, M.A.

[*Read Nov. 8th*, 1888.]

Raabe has given the name Bernoullian to the function

$$\phi(z, m) = z^m + m_1 B_0 z^{m-1} + m_2 B_1 z^{m-2} + m_3 B_2 z^{m-3} + \dots,$$

where there is no term without z; $m_1, m_2, \dots$ being the successive Binomial coefficients after the first in the expansion of $(1+t)^m$, and $B_0, B_1, \dots$ being Bernoulli's numbers, viz.,

$$B_0 = -\tfrac{1}{2},\quad B_2 = B_4 = B_6 = \dots = B_{2n} = \dots = 0,$$

$$B_1 = \tfrac{1}{6},\quad B_3 = -\tfrac{1}{30},\quad B_5 = \tfrac{1}{42},\quad B_7 = -\tfrac{1}{30},\quad B_9 = \tfrac{5}{66},\ \text{\&c.}$$

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[1] Colebroke, "Algebra," etc. (London, 1817), p. xix.
[2] De Morgan, "Arithmetical Books" (London, 1847), p. 47.
[3] Josephus, "Antiquities of the Jews" (Burder's Translation, Vol. I., pp. 616, 617). Burder's Preface is dated London, October 1, 1811.
[4] T. L. Heath, "Diophantos of Alexandria: a Study in the History of Greek Algebra" (Cambridge University Press, 1885).

For instance, the first Bernoullians are

$$\begin{aligned}
\phi(z, 1) &= z, \\
\phi(z, 2) &= z^2 - z, \qquad\qquad = u \text{ (say)}, \\
\phi(z, 3) &= z^3 - \tfrac{3}{2}z^2 + \tfrac{1}{2}z, \\
\phi(z, 4) &= z^4 - 2z^3 + z^2, \\
\phi(z, 5) &= z^5 - \tfrac{5}{2}z^4 + \tfrac{5}{3}z^3 - \tfrac{1}{6}z, \\
\phi(z, 6) &= z^6 - 3z^5 + \tfrac{5}{2}z^4 - \tfrac{1}{2}z^2, \\
&\&c. \qquad \&c.
\end{aligned}$$

Jacobi has shown [*Crelle*, Bd. XII. (1834), p. 271], and it is already well known, that

$$\begin{aligned}
\phi(z, 2n-1) &= \phi(z, 3) \,.\, F(u), \\
\phi(z, 2n) &= \phi(z, 4) \,.\, G(u),
\end{aligned}$$

where $F(u)$, $G(u)$ are functions of u of degree $n-2$. It might, therefore, be inferred that $F(u)$ and $G(u)$ might be expressed in terms of Bernoullians not higher than the $(2n-4)$th. The object of the present paper is to prove this and find the development.

Writing

$$\phi(z, 2n-1) = \phi(z, 3) \left\{ a_0 z^{2n-4} + a_1 z^{2n-5} + \ldots + a_r z^{2n-r-4} + \ldots + a_{2n-4} \right\} \qquad \ldots\ldots\ldots\ldots\ldots(1),$$

and noting that

$$\begin{aligned}
\phi(z, 3) &= z^3 - \tfrac{3}{2}z^2 + \tfrac{1}{2}z, \\
\phi(z, 2n-1) &= z^{2n-1} + (2n-1)\, B_0 z^{2n-2} + (2n-1)_2 B_1 z^{2n-3} \\
&\qquad + (2n-1)_3 B_2 z^{2n-4} + \ldots
\end{aligned}$$

on expanding the right-hand side of (1), and equating coefficients of powers of z, we get the equations

$$\begin{aligned}
a_r - \tfrac{3}{2}a_{r-1} + \tfrac{1}{2}a_{r-2} &= (2n-1)_r \;\; B_{r-1}, \\
a_{r-1} - \tfrac{3}{2}a_{r-2} + \tfrac{1}{2}a_{r-3} &= (2n-1)_{r-1} B_{r-2}, \\
\ldots \quad \ldots \quad \ldots \quad & \ldots \quad \ldots \quad \ldots \\
a_2 - \tfrac{3}{2}a_1 + \tfrac{1}{2}a_0 &= (2n-1)_2 B_1, \\
a_1 - \tfrac{3}{2}a_0 &= (2n-1)\;\; B_0, \\
a_0 &= 1.
\end{aligned}$$

Hence, the value of a_r may be expressed as a determinant of the $(r+1)^{\text{th}}$ order, viz., after a slight simplification,

$$2^r a_r = \begin{vmatrix} 3, & 1, & ., & . & \dots\dots, & \pm(2n-1)_r\ B_{r-1} \\ 2, & 3, & 1, & . & \dots\dots, & \mp(2n-1)_{r-1} B_{r-2} \\ ., & 2, & 3, & 1 & \dots\dots, & \pm(2n-1)_{r-2} B_{r-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots\dots & 2, & 3, & 1, & ., & -(2n-1)_3 B_2 \\ \dots\dots & ., & 2, & 3, & 1, & (2n-1)_2 B_1 \\ \dots\dots & ., & ., & 2, & 3, & -(2n-1)\ B_0 \\ \dots\dots & ., & ., & ., & 2, & 1 \end{vmatrix} (r+1)^{\text{th}} \text{ order,}$$

the signs of the terms in the last column being alternately + and −, beginning with the lowest term as +. This is easily evaluated, and gives

$$2^r a_r = (2^{r+1}-1)+2\,(2^r-1)(2n-1)\,B_0+2^2\,(2^{r-1}-1)(2n-1)_2 B_1$$
$$+2^3\,(2^{r-2}-1)(2n-1)_3 B_2+\dots$$
$$\dots+2^{r-1}\,(2^2-1)(2n-1)_{r-1} B_{r-2}+2^r\,(2n-1)_r B_{r-1}\dots(2).$$

Similarly, writing

$$\phi\,(z, 2n) = \phi\,(z, 4)\,\left\{b_0 z^{2n-4}+b_1 z^{2n-5}+\dots+b_r z^{2n-r-4}+\dots+b_{2n-4}\right\}\dots(3);$$

and noting that

$$\phi\,(z, 4) \;= z^4-2z^3+z^2,$$
$$\phi\,(z, 2n) = z^{2n}+2nB_0 z^{2n-1}+(2n)_2 B_1 z^{2n-2}+(2n)_3 B_2 z^{2n-3}+\dots;$$

on expanding the right-hand side of (3), and equating coefficients of powers of z, we get the equations

$$b_r-2b_{r-1}+b_{r-2} = (2n)_r B_{r-1},$$
$$\vdots \qquad\qquad \vdots$$
$$b_2-\;2b_1\;+\;b_0\; = (2n)_2 B_1,$$
$$b_1\;-\;2b_0 = (2n)\;B_0,$$
$$b_0 = 1,$$

whence, as before,

$$b_r = \begin{vmatrix} 2, & 1, & ., & \ldots\ldots\ldots\ldots, & & \pm(2n)_r \ B_{r-1} \\ 1, & 2, & 1, & \ldots\ldots\ldots\ldots, & & \mp(2n)_{r-1} B_{r-2} \\ ., & 1, & 2, & \ldots\ldots\ldots\ldots, & & \pm(2n)_{r-2} B_{r-3} \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots & \ldots\ldots \\ \ldots\ldots & 1, & 2, & 1, & ., & -(2n)_3 B_2 \\ \ldots\ldots & ., & 1, & 2, & 1, & (2n)_2 B_1 \\ \ldots\ldots & ., & ., & 1, & 2, & -(2n) \ B_0 \\ \ldots\ldots & ., & ., & ., & 1, & 1 \end{vmatrix} \ (r+1)^{\text{th}} \text{ order},$$

the signs of the terms in the last column being again alternately + and −, commencing with the lowest as +, and reading upwards.

Finally,

$$b_r = (r+1) + r\,(2n)\,B_0 + (r-1)\,(2n)_2\,B_1 + (r-2)\,(2n)_3\,B_2 + \ldots$$
$$\ldots + 2\,(2n)_{r-1}\,B_{r-2} + 1\,(2n)_r\,B_{r-1} \ldots\ldots\ldots\ldots (4).$$

Now assume

$$\phi\,(z, 2n-1) = \phi\,(z, 3)\,\{c_0\phi\,(z, 2n-4) + c_1\phi\,(z, 2n-5) + \ldots$$
$$\ldots + c_{2n-6}\,\phi\,(z, 2) + c_{2n-5}\,\phi\,(z, 1) + c_{2n-4}\},$$

$$\phi\,(z, 2n) = \phi\,(z, 4)\,\{d_0\phi\,(z, 2n-4) + d_1\phi\,(z, 2n-5) + \ldots$$
$$\ldots + d_{2n-6}\phi\,(z, 2) + d_{2n-5}\phi\,(z, 1) + d_{2n-4}\};$$

it is required to find the connections between the a's and the c's, and between the b's and the d's. To do this it will be sufficient to take a special case, as the method is general and the result simple. As soon as we commence the investigation of these connections, on remembering that the *even* Bernoulli's numbers vanish, we find that all the *odd* c's and d's vanish. Taking advantage of this simplification, let, for example,

$$\phi\,(z, 13) = \phi\,(z, 3)\,\{c_0\phi\,(z, 10) + c_2\phi\,(z, 8) + c_4\phi\,(z, 6) + c_6\phi\,(z, 4)$$
$$+ c_8\phi\,(z, 2) + c_{10}\},$$

then the coefficient of $\phi\,(z, 3)$ may be expanded in powers of z and

tabulated thus—

	z^{10}	z^9	z^8	z^7	z^6	z^5	z^4	z^3	z^2	z	1
$c_0\phi(z, 10)$	c_0	$c_0 10_1 B_0$	$c_0 10_2 B_1$		$c_0 10_4 B_3$		$c_0 10_6 B_5$		$c_0 10_8 B_7$		
$c_2\phi(z, 8)$			c_2	$c_2 8_1 B_0$	$c_2 8_2 B_1$		$c_2 8_4 B_3$		$c_2 8_6 B_5$		
$c_4\phi(z, 6)$					c_4	$c_4 6_1 B_0$	$c_4 6_2 B_1$		$c_4 6_4 B_3$		
$c_6\phi(z, 4)$							c_6	$c_6 4_1 B_0$	$c_6 4_2 B_1$		
$c_8\phi(z, 2)$									c_8	$c_8 2_1 B_0$	
c_{10}											c_{10}

a function which is to be identical with the coefficient of $\phi(z, 3)$ in equation (1), when $n = 7$.

Hence $$c_0 10_1 B_0 = a_1 \quad \text{or} \quad c_0 = -\frac{a_1}{5} = a_0 = 1,$$

$$c_2\, 8_1\, B_0 = a_3 \quad \text{or} \quad c_2 = -\frac{a_3}{4},$$

$$c_4\, 6_1\, B_0 = a_5 \quad \text{or} \quad c_4 = -\frac{a_5}{3},$$

$$c_6\, 4_1\, B_0 = a_7 \quad \text{or} \quad c_6 = -\frac{a_7}{2},$$

$$c_8\, 2_1\, B_0 = a_9 \quad \text{or} \quad c_8 = -\frac{a_9}{1},$$

and $$c_{10} = a_{10}.$$

In a similar manner, taking the Bernoullians of even degree, and for example, assuming

$$\phi(z, 14) = \phi(z, 4)\,\{d_0\phi(z, 10) + d_2\phi(z, 8) + d_4\phi(z, 6) + d_6\phi(z, 4) + d_8\phi(z, 2) + d_{10}\},$$

we shall find

$$d_0 10_1 B_0 = b_1 \quad \text{or} \quad d_0 = -\frac{b_1}{5} = b_0 = 1,$$

$$d_2\, 8_1\, B_0 = b_3 \quad \text{or} \quad d_2 = -\frac{b_3}{4},$$

$$d_4\, 6_1\, B_0 = b_5 \quad \text{or} \quad d_4 = -\frac{b_5}{3},$$

$$d_6\, 4_1\, B_0 = b_7 \quad \text{or} \quad d_6 = -\frac{b_7}{2},$$

$$d_8\, 2_1\, B_0 = b_9 \quad \text{or} \quad d_8 = -\frac{b_9}{1},$$

$$d_{10} = b_{10}.$$

Examples—

$$\frac{\phi(z,13)}{\phi(z,3)} = z^{10} - 5z^9 + 5z^8 + 10z^7 - \frac{34}{3}z^6 - 22z^5 + \frac{284}{21}z^4$$

$$+ \frac{219}{7}z^3 - \frac{41}{15}z^2 - \frac{691}{35}z - \frac{691}{105},$$

$$= \phi(z,10) - \frac{5}{2}\phi(z,8) + \frac{22}{3}\phi(z,6) - \frac{219}{14}\phi(z,4) + \frac{691}{35}\phi(z,2) - \frac{691}{105};$$

$$\frac{\phi(z,14)}{\phi(z,4)} = z^{10} - 5z^9 + \frac{25}{6}z^8 + \frac{40}{3}z^7 - \frac{163}{15}z^6 - \frac{526}{15}z^5 + \frac{367}{30}z^4$$

$$+ \frac{893}{15}z^3 + \frac{101}{15}z^2 - \frac{691}{15}z - \frac{691}{30}$$

$$= \phi(z,10) - \frac{10}{3}\phi(z,8) + \frac{526}{45}\phi(z,6) - \frac{893}{30}\phi(z,4) + \frac{691}{15}\phi(z,2) - \frac{691}{30}.$$

The theorem is finally stated thus, ϵ being 3 or 4 as the case may be :

If

$$\phi(z, 2m+\epsilon) = \phi(z,\epsilon)\left\{e_0 z^{2m} + e_1 z^{2m-1} + e_2 z^{2m-2} + \ldots + e_{2m-2}z^2 + e_{2m-1}z + e_{2m}\right\},$$

where, when $\epsilon = 3$,

$$e_r = \frac{2^{r+1}-1}{2^r} + \frac{2^r-1}{2^{r-1}}(2m+\epsilon)B_0 + \frac{2^{r-1}-1}{2^{r-2}}(2m+\epsilon)_2 B_1$$

$$+ \frac{2^{r-2}-1}{2^{r-3}}(2m+\epsilon)_3 B_2 + \ldots,$$

and when $\epsilon = 4$,

$$e_r = (r+1) + r(2m+\epsilon)B_0 + (r-1)(2m+\epsilon)_2 B_1 + (r-2)(2m+\epsilon)_3 B_2 + \ldots$$

then $$\phi(z, 2m+\epsilon) = \phi(z,\epsilon)\left\{\phi(z,2m) - \frac{e_3}{m-1}\phi(z, 2m-2)\right.$$

$$\left. - \frac{e_5}{m-2}\phi(z,2m-4) - \ldots - \frac{e_{2m-3}}{2}\phi(z,4) - \frac{e_{2m-1}}{1}\phi(z,2) + e_{2m}\right\}.$$

The first 14 Bernoullians expressed in terms of lower Bernoullians are as follows :—

$$\phi(z,5) = \phi(z,3)\left\{\phi(z,2) - \frac{1}{3}\right\}$$

$$\phi(z,6) = \phi(z,4)\left\{\phi(z,2) - \frac{1}{2}\right\},$$

$$\phi(z,7) = \phi(z,3)\left\{\phi(z,4) - \phi(z,2) + \frac{1}{3}\right\},$$

$$\phi(z,8) = \phi(z,4)\left\{\phi(z,4) - \frac{4}{3}\phi(z,2) + \frac{2}{3}\right\},$$

$$\phi(z,9) = \phi(z,3)\left\{\phi(z,6) - \frac{3}{2}\phi(z,4) + \frac{9}{5}\phi(z,2) - \frac{3}{5}\right\},$$

$$\phi(z,10) = \phi(z,4)\left\{\phi(z,6) - 2\phi(z,4) + 3\phi(z,2) - \frac{3}{2}\right\},$$

$$\phi(z,11) = \phi(z,3)\left\{\phi(z,8) - 2\phi(z,6) + 4\phi(z,4) - 5\phi(z,2) + \frac{5}{3}\right\},$$

$$\phi(z,12) = \phi(z,4)\left\{\phi(z,8) - \frac{8}{3}\phi(z,6) + \frac{13}{2}\phi(z,4) - 10\phi(z,2) + 5\right\},$$

$$\phi(z,13) = \phi(z,3)\left\{\phi(z,10) - \frac{5}{2}\phi(z,8) + \frac{22}{3}\phi(z,6) - \frac{219}{14}\phi(z,4) + \frac{691}{35}\phi(z,2) - \frac{691}{105}\right\},$$

$$\phi(z,14) = \phi(z,4)\left\{\phi(z,10) - \frac{10}{3}\phi(z,8) + \frac{526}{45}\phi(z,6) - \frac{893}{30}\phi(z,4) + \frac{691}{15}\phi(z,2) - \frac{691}{30}\right\}.$$

We may note that the a's and b's are connected by the relation

$$(2n-r)\,b_r - (2n-r-1)\,b_{r-1} = 2na_r - na_{r-1} \quad \ldots\ldots\ldots\ldots(5).$$

For, by obvious steps, it is easily seen that ($m = 2n$)

$$b_r = \text{coefficient of } z^r \text{ in } \frac{z^{r+1}-1}{z-1}\frac{d}{dz}\left\{\frac{\phi(z,m)}{z^{m-r-1}}\right\},$$

$$= \quad ,, \quad z^m \text{ in } \frac{z^{r+1}-1}{z-1}\left\{mz\phi(z,m-1)-(m-r-1)\phi(z,m)\right\},$$

$$= \quad ,, \quad z^m \text{ in } mz(z^{r+1}-1)z(z-\tfrac{1}{2})\frac{\phi(z,m-1)}{\phi(z,3)}$$

$$-(m-r-1)(z^{r+1}-1)z^2(z-1)\frac{\phi(z,m)}{\phi(z,4)},$$

$$= \quad ,, \quad z^m \text{ in } (mz^{r+4}-\tfrac{1}{2}mz^{r+3}-mz^3+\tfrac{1}{2}mz^2)$$

$$\times(\ldots+a_{r-1}z^{m-r-3}+a_rz^{m-r-4}+\ldots)$$

$$-(m-r-1)(z^{r+4}-z^{r+3}-z^3+z^2)(\ldots+b_{r-1}z^{m-r-3}+b_rz^{m-r-4}+\ldots),$$

therefore $b_r = ma_r-\tfrac{1}{2}ma_{r-1}-(m-r-1)b_r+(m-r-1)b_{r-1}$,

whence the required result, viz.,

$$(2n-r)b_r-(2n-r-1)b_{r-1}-2na_r+na_{r-1}=0.$$

It is interesting to observe that this equation is true, not on account of any relation among Bernoulli's numbers, but because the coefficient of each Bernoulli's number vanishes. Thus the term containing B_p in this equation is

$$\Big\{(2n-r)(r-p)(2n)_{p+1}-(2n-r-1)(r-p-1)(2n)_{p+1}$$

$$-2n\frac{2^{r-p}-1}{2^{r-p-1}}(2n-1)_{p+1}+n\frac{2^{r-p-1}-1}{2^{r-p-2}}(2n-1)_{p+1}\Big\}B_p,$$

which may be written in the form

$$\frac{(2n-p-1)(2n)_{p+1}}{2^{r-p-1}}B_p(2^{r-p-1}-2^{r-p}+1+2^{r-p-1}-1),$$

an expression which vanishes identically.

In fact, equation (5) is merely the arithmetical equivalent of the differential identity

$$\frac{d}{dz}\left\{\frac{\phi(z,2n)}{\phi(z,4)}\right\}+4\frac{\phi(z,3)}{\phi(z,4)}\frac{\phi(z,2n)}{\phi(z,4)}=2n\frac{\phi(z,2n-1)}{\phi(z,4)}.$$

On Deep-water Waves resulting from a Limited Original Disturbance. Mr. W. Burnside.

[*Read Nov. 8th*, 1888.]

Cauchy,* in his "Memoire sur la Théorie des Ondes," determines under certain limitations the disturbance at any point of the surface of deep water and at any time, due to an initial surface displacement over a given limited area.

Reckoning t from the time of the original disturbance and x along the surface from its centre, his result, when the motion is in two dimensions, is that the displacement

$$\propto \frac{t}{x^{\frac{3}{2}}} \sin\left(\frac{gt^2}{4x}+\frac{\pi}{4}\right).$$

The limitations mentioned are that l/x and $gt^2l/4x^2$ should both be very small quantities, l being the length of that part of the surface in which the original displacement is sensible.

The latter limitation robs the result of most of its interest, for it will be seen later that it is not until $gt^2l/4x^2$ has a finite value (in the particular case first treated it is $\frac{1}{2}$) for a given value of x that the displacement at the corresponding place reaches its maximum.

Cauchy's expression is obtained as follows:

If the original vertical displacement at any point (the motion being supposed to start from rest) is $f(x)$, the form of the surface at any subsequent time is given by

$$y=\frac{1}{\pi}\int_0^\infty dm\int_{-\infty}^\infty da \cos m^{\frac{1}{2}}g^{\frac{1}{2}}t \cos m\,(a-x)\,f(a).$$

If $f(a)$ vanishes for all but very small values of a,

$$\int_{-\infty}^\infty \cos m\,(a-x)\,f(a)\,da = H\cos mx \text{ approximately,}$$

where H is some constant quantity. (Cauchy shews that it is here that the limitation of $gt^2l/4x^2$ to small values is introduced.)

Therefore $$y=\frac{H}{\pi}\int_0^\infty \cos m^{\frac{1}{2}}g^{\frac{1}{2}}t \cos mx\,dm.$$

* *Mém. des Sav. Etran.*, Vol. I., 1827.

Next, the value of the integral

$$\frac{1}{\pi}\int_0^\infty e^{-am}\cos m^{\frac{1}{2}}g^{\frac{1}{2}}t\cos mx\,dm$$

is calculated in the form of an infinite series, and in the result zero is written for a. If this is called I, then finally

$$y = HI.$$

It will be seen that, though he starts from a perfectly general form for the original surface displacement, Cauchy, as a matter of fact, introduces in the course of his work a special form, and that his result holds only for the latter under certain conditions.

Indeed his final equation may be written

$$y = \frac{H}{\pi}L^t_{a=0}\int_0^\infty e^{-am}\cos m^{\frac{1}{2}}g^{\frac{1}{2}}t\cos mx\,dm,$$

and hence the initial surface displacement is

$$y_0 = \frac{H}{\pi}L^t_{a=0}\int_0^\infty e^{-am}\cos mx\,dm$$

$$= \frac{H}{\pi}L^t_{a=0}\frac{a}{a^2+x^2}.$$

Strictly, then, the case investigated is one in which the original displacement $\propto 1/a^2+x^2$, and the result is obtained for such times and places that zero may be written for a at a certain stage without sensible error.

In what follows I have attempted, starting first from the same initial form, but retaining the quantity a throughout, to obtain an expression for the displacement, which at a given place, shall hold through a much greater range of time than Cauchy's, and in particular from which it may be possible to determine when the displacement at a given place is a maximum and also its magnitude.

Taking c as the greatest initial displacement of any point of the surface, and l as the length of surface over which the initial disturbance is sensible, I shall speak of a term $c\,(l/x)^n$ in the expression for the displacement at any point as of the order $(l/x)^n$. The displacement will be only considered for points for which l/x is a small quantity.

Putting, then,
$$y_0 = ca\int_0^\infty e^{-am}\cos mx\,dm$$

$$= \frac{ca^2}{a^2+x^2},$$

the initial displacement at any distant point is of the order $(l/x)^2$, and this is to be considered vanishingly small.

The theory of deep-water waves shews that, if the surface be initially displaced but motionless, the typical solution is

$$y = A\cos mx \cos m^{\frac{1}{2}}g^{\frac{1}{2}}t,$$

the corresponding initial form being

$$y = A\cos mx;$$

and by an application of Fourier's theorem, any arbitrary form of free surface may be built up from this typical solution.

Hence at once the form of the surface at time t, corresponding to the above value y_0, for the initial displacement, is

$$y = ca\int_0^\infty e^{-am}\cos mx \cos m^{\frac{1}{2}}g^{\frac{1}{2}}t\,dm.$$

By expanding the term $\cos m^{\frac{1}{2}}g^{\frac{1}{2}}t$, and integrating term by term, this becomes

$$y = \frac{ca}{(a^2+x^2)^{\frac{1}{2}}}\sum_0^\infty \frac{(-1)^n n!}{2n!}\frac{(gt^2)^n}{(a^2+x^2)^{\frac{1}{2}n}}\cos\left(\overline{n+1}\tan^{-1}\frac{x}{a}\right),$$

or, if
$$\frac{gt^2}{(a^2+x^2)^{\frac{1}{2}}} = 2z, \quad \tan^{-1}\frac{x}{a} = \beta,$$

$$y = c\cos\beta\left[\cos\beta - z\cos 2\beta + \frac{z^2}{1.3}\cos 3\beta - \frac{z^3}{1.3.5}\cos 4\beta + \ldots\right].$$

Now, it may be easily verified, by integrating by parts, that

$$2\cos\beta\left[z\cos 2\beta - \frac{z^2}{1.3}\cos 3\beta + \frac{z^3}{1.3.5}\cos 4\beta - \ldots\right]$$
$$= (z\cos\beta)^{\frac{1}{2}}\int_0^{z\cos\beta}\frac{e^{-y/2}\cos(2\beta - \frac{1}{2}y\tan\beta)}{(z\cos\beta - y)^{\frac{1}{2}}}\,dy,$$

and hence
$$y = c\cos^2\beta - \tfrac{1}{2}c\,(z\cos\beta)^{\frac{1}{2}}\int_0^{z\cos\beta}\frac{e^{-y/2}\cos(2\beta - \frac{1}{2}y\tan\beta)}{(z\cos\beta - y)^{\frac{1}{2}}}\,dy.$$

If in the integral the variable be changed by the substitution

$$y = z\cos\beta - 2y'\cot\beta,$$

we have, after reduction,

$$y = c\cos^2\beta - c\cos\beta\left(\frac{z}{2\sin\beta}\right)^{\frac{1}{2}} e^{-\frac{1}{2}z\cos\beta}$$

$$\times\left[\cos\left(2\beta-\frac{z\sin\beta}{2}\right)\int_0^{\frac{1}{2}z\sin\beta}\frac{e^{y'\cot\beta}\cos y'}{y'^{\frac{1}{2}}}\,dy'\right.$$

$$\left.-\sin\left(2\beta-\frac{z\sin\beta}{2}\right)\int_0^{\frac{1}{2}z\sin\beta}\frac{e^{y'\cot\beta}\sin y'}{y'^{\frac{1}{2}}}\,dy'\right].$$

The integrals now to be evaluated are of the form

$$\int_0^p \frac{e^{qx}\begin{matrix}\cos\\ \sin\end{matrix}x}{x^{\frac{1}{2}}}\,dx,$$

where q is a very small numerical quantity. By expanding the exponential and integrating each term by parts, the following expressions are at once obtained for the two integrals:—

$$\int_0^p \frac{e^{qx}\cos x}{x^{\frac{1}{2}}}\,dx = \left(1-\frac{1\,.\,3}{2\,.\,4}q^2+\frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}q^4-\dots\right)\int_0^p\frac{\cos x}{x^{\frac{1}{2}}}\,dx$$

$$+\left(-\frac{1}{2}q+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}q^3-\frac{1\,.\,3\,.\,5\,.\,7\,.\,9}{2\,.\,4\,.\,6\,.\,8\,.\,10}q^5+\dots\right)\int_0^p\frac{\sin x}{x^{\frac{1}{2}}}\,dx$$

$$+\frac{\sin p}{p^{\frac{1}{2}}}\left(pq+\frac{p^2q^2}{2!}+\frac{p^3q^3}{3!}+\dots\right)$$

$$+\frac{\cos p}{p^{\frac{3}{2}}}\left(\frac{3}{2}\,\frac{p^2q^2}{2!}+\frac{5}{2}\,\frac{p^3q^3}{3!}+\frac{7}{2}\,\frac{p^4q^4}{4!}+\dots\right)$$

$$-\frac{\sin p}{p^{\frac{5}{2}}}\left(\frac{3\,.\,5}{2\,.\,2}\,\frac{p^3q^3}{3!}+\frac{5\,.\,7}{2\,.\,2}\,\frac{p^4q^4}{4!}+\dots\right)$$

$$-\frac{\cos p}{p^{\frac{7}{2}}}\left(\frac{3\,.\,5\,.\,7}{2\,.\,2\,.\,2}\,\frac{p^4q^4}{4!}+\frac{5\,.\,7\,.\,9}{2\,.\,2\,.\,2}\,\frac{p^5q^5}{5!}+\dots\right)+\text{etc.},$$

$$\int_0^p \frac{e^{qx}\sin x}{x^{\frac{1}{2}}}\,dx = \left(1-\frac{1\,.\,3}{2\,.\,4}q^2+\frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}q^4-\dots\right)\int_0^p\frac{\sin x}{x^{\frac{1}{2}}}\,dx$$

$$+\left(\frac{1}{2}q-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}q^3+\dots\right)\int_0^p\frac{\cos x}{x^{\frac{1}{2}}}\,dx$$

$$-\frac{\cos p}{p^{\frac{1}{2}}}\left(pq+\frac{p^2q^2}{2!}+\dots\right)$$

$$+\frac{\sin p}{p^{\frac{3}{2}}}\left(\frac{3}{2}\,\frac{p^2q^2}{2!}+\frac{5}{2}\,\frac{p^3q^3}{3!}+\dots\right)$$

$$+\frac{\cos p}{p^{\frac{5}{2}}}\left(\frac{3\,.\,5}{2\,.\,2}\,\frac{p^3q^3}{3!}+\frac{5\,.\,7}{2\,.\,2}\,\frac{p^4q^4}{4!}+\dots\right)$$

$$-\frac{\sin p}{p^{\frac{7}{2}}}\left(\frac{3\,.\,5\,.\,7}{2\,.\,2\,.\,2}\,\frac{p^4q^4}{4!}+\frac{5\,.\,7\,.\,9}{2\,.\,2\,.\,2}\,\frac{p^5q^5}{5!}+\dots\right)-\text{etc.}$$

The general term of the part occurring after the first two lines in either of these integrals is

$$\pm \frac{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix} p}{p^{\frac{1}{2}(2n-1)}} \left[\frac{2n-1 \, . \, 2n-3 \ldots 3}{2^{n-1}} \, \frac{(pq)^n}{n!} + \frac{\overline{2n+1} \ldots 5}{2^{n-1}} \, \frac{(pq)^{n+1}}{n+1!} + \ldots \right],$$

and the quantity in brackets is clearly less than

$$(pq)^{n-1} \, (e^{pq} - 1).$$

The general term of the series is therefore in magnitude less than

$$\frac{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix} p}{p^{\frac{1}{2}}} \, (e^{pq} - 1) \, q^{n-1}.$$

Hence, q being very small, we have as first approximations

$$\int_0^p \frac{e^{qx} \cos x}{x^{\frac{1}{2}}} \, dx = \int_0^p \frac{\cos x}{x^{\frac{1}{2}}} \, dx + \frac{\sin p}{p^{\frac{1}{2}}} \, (e^{pq} - 1),$$

$$\int_0^p \frac{e^{qx} \sin x}{x^{\frac{1}{2}}} \, dx = \int_0^p \frac{\sin x}{x^{\frac{1}{2}}} \, dx - \frac{\cos p}{p^{\frac{1}{2}}} \, (e^{pq} - 1),$$

the next terms of either part of either integral being of the order q times those written.

Also $(e^{pq}-1)/p^{\frac{1}{2}}$ increases continually with p, and for moderate values of pq (which will be found to correspond to the case here to be considered) is of the order $q^{\frac{1}{2}}$, and therefore in the expression of each integral the second terms written are intermediate in magnitude between the first and those omitted.

Finally, as is well known,

$$\int_0^p \frac{\cos x}{x^{\frac{1}{2}}} \, dx = \sqrt{\frac{\pi}{2}} + \frac{\sin p}{p^{\frac{1}{2}}} - \frac{1}{2} \, \frac{\cos p}{p^{\frac{3}{2}}} - \frac{1 \, . \, 3}{2 \, . \, 2} \, \frac{\sin p}{p^{\frac{5}{2}}} + \ldots,$$

$$\int_0^p \frac{\sin x}{x^{\frac{1}{2}}} \, dx = \sqrt{\frac{\pi}{2}} - \frac{\cos p}{p^{\frac{1}{2}}} - \frac{1}{2} \, \frac{\sin p}{p^{\frac{3}{2}}} + \frac{1 \, . \, 3}{2 \, . \, 2} \, \frac{\cos p}{p^{\frac{5}{2}}} + \ldots,$$

where the divergent series may be used as approximations by stopping at or before the smallest terms.

Before substituting in the expression for the displacement the values of the integrals now found, it will be well to consider the magnitudes of some of the quantities involved, with a view to neglecting systematically terms q times those retained.

Thus $\cos\left(2\beta - \frac{z \sin \beta}{2}\right)$ and $\sin\left(2\beta - \frac{z \sin \beta}{2}\right)$

differ from $$-\cos\left(\frac{z\sin\beta}{2}\right) \text{ and } \sin\left(\frac{z\sin\beta}{2}\right),$$

by quantities of the order q times these latter quantities; also

$$p = \frac{z\sin\beta}{2} = \frac{gt^2x}{4(x^2+a^2)},$$

and hence, except in the circular functions $gt^2/4x$, may certainly be written for p; so also

$$\frac{z\cos\beta}{2}\,(=pq) = \frac{gt^2a}{4x^2}, \quad \cos\beta = \frac{a}{x},$$

and $$\frac{z}{2\sin\beta} = \frac{gt^2}{4x} \text{ approximately.}$$

Hence, retaining p for $\frac{1}{2}z\sin\beta$ in the circular functions, we get

$$y = \frac{ca}{x}\left(\frac{gt^2}{4x}\right)^{\frac{1}{2}} e^{-gt^2a/4x^2}$$

$$\times\left[\cos p\left\{\sqrt{\frac{\pi}{2}} + \left(\frac{gt^2}{4x}\right)^{-\frac{1}{2}} e^{gt^2a/4x^2}\sin p - \frac{1}{2}\left(\frac{gt^2}{4x}\right)^{-\frac{3}{2}}\cos p - \text{etc.}\right\}\right.$$

$$\left.+\sin p\left\{\sqrt{\frac{\pi}{2}} - \left(\frac{gt^2}{4x}\right)^{-\frac{1}{2}} e^{gt^2a/4x^2}\cos p - \frac{1}{2}\left(\frac{gt^2}{4x}\right)^{-\frac{3}{2}}\sin p + \text{etc.}\right\}\right]$$

$$= \frac{ca}{x}\left(\frac{gt^2}{4x}\right)^{\frac{1}{2}} e^{-gt^2a/4x^2}$$

$$\times\left[\sqrt{\pi}\sin\left(p+\frac{\pi}{4}\right) - \frac{1}{2}\left(\frac{gt^2}{4x}\right)^{-\frac{3}{2}} + \frac{1.3.5}{2.2.2}\left(\frac{gt^2}{4x}\right)^{-\frac{7}{2}} - \dots\right].$$

With the same coefficient outside the bracket, the most important of the omitted periodic terms inside is

$$\frac{3\sqrt{\pi}}{2}\,\frac{a}{x}\cos\left(p+\frac{\pi}{4}\right).$$

The ultimately divergent series following the first term inside the bracket, may, as in the case of the two from which it is derived, be used as an approximation by stopping before the smallest term; and, as it converges for three terms if $gt^2/4x$ is greater than 4, the above expression will serve for the calculation of y at a given point for any values of the time not less than $4\sqrt{x/g}$.

For small values of t the original series is certainly the best form from which to calculate y; and from it may be seen that for such values y is of the order $(a/x)^2$, as it is initially. The expression for y now found shows that, when $gt^2/4x$ is a moderately large number, y is of the

order a/x; but also indicates that, when $gt^2/4x$ is comparable with x/a, y is of the order $\sqrt{a/x}$. For values of t such as this all terms following $\sqrt{\pi}\sin(p+\pi/4)$ may certainly be omitted as smaller than terms already neglected. Moreover, when $gt^2/4x$ is comparable with x^2/a^2, the value of y will, owing to the factor $e^{-gt^2a/4x^2}$, be again extremely small, and hence in the term $\sin(p+\pi/4)$ we may safely write $gt^2/4x$ for p. We have finally then, as an expression for the displacement at a given place, as long as its magnitude is comparable with its maximum value,

$$y = \sqrt{\pi}\,\frac{ca}{x}\left(\frac{gt^2}{4x}\right)^{\frac{1}{2}} e^{-gt^2a/4x^2} \sin\left(\frac{gt^2}{4x}+\frac{\pi}{4}\right).$$

The greatest value of

$$\left(\frac{gt^2}{4x}\right)^{\frac{1}{2}} e^{-gt^2a/4x^2}$$

for a particular value of x, is given by

$$gt^2a = 2x^2,$$

and the ratio of the logarithmic differential coefficient of $\left(\frac{gt^2}{4x}\right)^{\frac{1}{2}} e^{-gt^2a/4x^2}$ to that of $gt^2/4x$ is

$$1-\frac{gt^2a}{2x^2};$$

hence, when, for a given value of x, y is near its greatest value, $\sin\left(\frac{gt^2}{4x}+\frac{\pi}{4}\right)$ changes very rapidly compared to its coefficient; or, in other words, to determine the maximum value of y, the coefficient only need be considered.

The greatest amplitude of displacement at the point x is therefore

$$\sqrt{\frac{\pi}{2}}\,ce^{-\frac{1}{2}}\left(\frac{a}{x}\right)^{\frac{1}{2}},$$

and the amplitude has this value when, as seen above,

$$t = x\sqrt{\frac{2}{ga}}.$$

Also, if $gt^2/4x$ increases by 2π between times t and t', then

$$t'-t = \frac{8\pi x}{g(t'+t)};$$

or, in words, the period of the wave motion at x, when the amplitude is greatest, is

$$2\pi\sqrt{\frac{2a}{g}},$$

and the length of the waves is $4\pi a$.

For places at which the displacement has not reached its maximum the period and wave-length are greater, and for those for which the greatest displacement is past they are less than these quantities.

The energy of the initial displacement varies as c^2a, and the extent of surface over which it is sensible varies as a. The above results may then be expressed as follows.

For the same *form* of initial displacement—

(i.) The greatest amplitude of displacement at any point varies as the square root of the whole energy of the motion, and inversely as the square root of the distance from the originally disturbed area.

(ii.) The greatest disturbance is propagated with a uniform velocity, varying as the square root of the space over which the original disturbance is sensible; and

(iii.) The wave-length of the motion, when greatest, varies as the space over which the original disturbance is sensible.

These results are proved only for the case now considered; but, from the nature of the problem, there can be little doubt but that similar statements may be made with regard to the motions resulting from any other *form* of initial displacement. Indeed this is shown to be the case later for two other forms.

To represent the results in a complete way graphically would involve very considerable labour; but, omitting the periodic term in the expression for the displacement, the magnitude of the motion only may be represented as follows.

If in the general expression for y the periodic factor be omitted, then, when x is regarded as constant,

$$y = Ate^{-\alpha t^2},$$

and, when t is regarded as constant,

$$y = \frac{B}{x^{\frac{3}{2}}}e^{-\beta/x^2},$$

where A, B, α, β are constant quantities. The general forms of the curves with these equations will not depend on the values of the constants, which affect only the relative scales parallel to the two axes on which they are drawn.

Hence the curves $\qquad y = 5xe^{-x^2}$ .... (i.),

$$y = \frac{5}{x^{\frac{3}{2}}}e^{-1/x^2} \quad \text{.............................(ii.),}$$

(where the numerical factor is introduced to give suitable propor-

tions to the figures,) will show graphically—the first, how the amplitude of the displacement varies with the time at a particular place; the second, how the amplitudes vary from place to place at a given time.

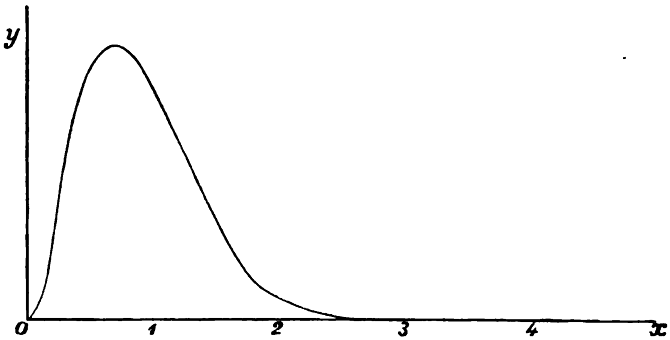

FIG. 1.

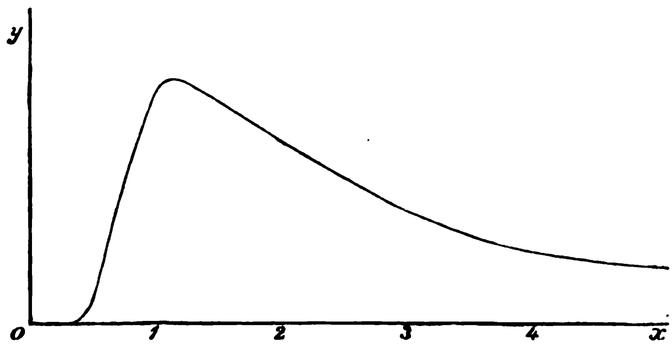

FIG. 2.

These curves enable us to further interpret the formula for y in a general way as follows.

At a particular point the disturbance dies away from its maximum value at a slower rate than it grows up to it; though, the first curve not being very unsymmetrical on either side of its maximum ordinate, the rise and dying away of the disturbance at a particular point will not be very dissimilar.

At a particular time, the greater part of the surface sensibly disturbed will be beyond the point of maximum disturbance, on the near side of which the amplitude rapidly diminishes to an insensible amount.

These results, which at first sight appear almost to contradict

each other, are reconciled at once when it is noticed that at any instant the extent of sensibly disturbed surface (*i.e.*, the amount of surface in which the amplitude of the disturbance is not less than a given sub-multiple of the existing greatest disturbance) increases in direct proportion with the distance of the point of greatest disturbance from the origin.

It appears certain that, for the particular form of original disturbance hitherto considered, y cannot be *exactly* expressed in terms of x and t otherwise than by an infinite series. It is not, however, difficult artificially to build up forms for the original disturbance, such that the value of y for any values of x and t can be expressed in finite terms; and if these are chosen so that the extent of surface over which the original disturbance is sensible is limited, the results will serve as a sort of test of the accuracy of the approximate value of y obtained above, and of the law of the motion deduced from it.

In particular, if

$$y_0 = A \int_0^\infty m^{i+\frac{1}{2}} e^{-am} \cos mx \, dm,$$

where i is a positive integer, the resulting value of y may be expressed in finite terms.

Taking the case of i zero, put

$$y_0 = \frac{2c}{\sqrt{\pi}} a^{\frac{3}{2}} \int_0^\infty m^{\frac{1}{2}} e^{-am} \cos mx \, dm$$

$$= c \left(\frac{a^2}{a^2+x^2}\right)^{\frac{3}{4}} \cos\left(\frac{3}{2} \tan^{-1} \frac{x}{a}\right).$$

Then $$y = \frac{2ca^{\frac{3}{2}}}{\sqrt{\pi}} \int_0^\infty m^{\frac{1}{2}} e^{-am} \cos mx \cos m^{\frac{1}{2}} g^{\frac{1}{2}} t \, dm.$$

Put $$mx = m'^2,$$

$$\tan^{-1} \frac{x}{a} = \beta,$$

$$\frac{gt^2}{4x} = p,$$

and therefore

$$y = \frac{4c}{\sqrt{\pi}} \left(\frac{a}{x}\right)^{\frac{3}{2}} \int_0^\infty m^2 e^{-m^2 \cot \beta} \cos m^2 \cos 2mp^{\frac{1}{2}} \, dm.$$

This integral may be deduced from the known form

$$\int_0^\infty e^{-qm^2}\cos 2mp^{\frac{1}{2}}\,dm = \frac{\sqrt{\pi}}{2}\,\frac{1}{q^{\frac{1}{2}}}\,e^{-p/q}.$$

Differentiate this with respect to q, giving

$$\int_0^\infty m^2 e^{-qm^2}\cos 2mp^{\frac{1}{2}}\,dm = \frac{\sqrt{\pi}}{4}\left[\frac{1}{q^{\frac{3}{2}}}-\frac{2p}{q^{\frac{5}{2}}}\right]e^{-p/q}.$$

Now, for q write $\cot\beta+i$, and equate real parts; then

$$\int_0^\infty m^2 e^{-m^2\cot\beta}\cos m^2\cos 2mp^{\frac{1}{2}}\,dm$$

$$= \text{real part of } \frac{\sqrt{\pi}}{4}\left[\sin^{\frac{3}{2}}\beta\, e^{-3i\beta/2}-2p\sin^{\frac{5}{2}}\beta\, e^{-5i\beta/2}\right]e^{-p\sin\beta(\cos\beta-i\sin\beta)}$$

$$= \frac{\sqrt{\pi}}{4}\sin^{\frac{3}{2}}\beta\, e^{-p\sin\beta\cos\beta}\left[\cos\left(p\sin^2\beta-\frac{3\beta}{2}\right)\right.$$

$$\left.-2p\sin\beta\cos\left(p\sin^2\beta-\frac{5\beta}{2}\right)\right].$$

[The substitution of a complex value for q may be justified as follows:—The integral

$$\int_0^\infty m^2 e^{-qm^2}\cos 2mp^{\frac{1}{2}}\,dm$$

is the sum of a series of integrals of the form

$$\int_0^\infty m^{2n} e^{-qm^2}\,dm.$$

The result of writing $\cot\beta+i$ for q is the same, neglecting a factor in the integrand independent of m, as that of writing

$$\left(\cos\frac{\beta}{2}+i\sin\frac{\beta}{2}\right)m' \text{ for } m\sqrt{q\sin\beta},$$

and since $\beta/2$ is necessarily less than $\pi/4$, the known theory shows at once that the result of the substitution is correct.]

Finally, then, neglecting terms of the order a/x multiplied by those retained, so as to compare this result directly with the former one,

$$y = c\left(\frac{a}{x}\right)^{\frac{1}{2}} e^{-gt^2a/4x^2}\left[\frac{gt^2}{2x}\sin\left(\frac{gt^2}{4x}+\frac{\pi}{4}\right)-\cos\left(\frac{gt^2}{4x}+\frac{\pi}{4}\right)\right],$$

and for values of y at any place comparable with its maximum value, the second term inside the bracket may be neglected in comparison with the first.

It will be clear, on inspection of the value of y just obtained, that, with the new form of initial disturbance now considered, the general laws of the motion are the same as those already given. When, however, the particular results for the two given forms are compared, it will be seen that the initial form seems to have a decided effect on the resulting motion.

Initial ordinate	$\dfrac{ca^2}{a^2+x^2}$	$c\left(\dfrac{a^2}{a^2+x^2}\right)^{\frac{3}{4}}\cos\left(\dfrac{3}{2}\tan^{-1}\dfrac{x}{a}\right)$
Whole energy of motion ...	$\dfrac{\pi}{4}gc^2a$	$\frac{1}{2}gc^2a$
Space in which initial ordinate diminishes to about $\frac{1}{100}$th of its greatest value ...	$10a$	$21a$
Maximum amplitude at place x ...	$\sqrt{\dfrac{\pi}{2}}\,ce^{-\frac{1}{2}}\left(\dfrac{a}{x}\right)^{\frac{1}{2}}$	$2ce^{-1}\left(\dfrac{a}{x}\right)^{\frac{1}{2}}$
Rate of propagation of greatest disturbance ...	$\sqrt{ga/2}$	$\sqrt{ga/4}$
Wave-length of motion when amplitude is greatest ...	$4\pi a$	$2\pi a$

The two cases hitherto considered are alike in this, that, the initial displacement being continuous, the extent of surface over which they are sensible can only be assigned by a convention as to what small quantities are to be considered negligible. It can hardly be doubted but that such a phenomenon as the Krakatao waves would be represented more nearly in symbols by a discontinuous initial form, the ordinate of which is strictly zero beyond a certain point. It is interesting, too, to notice how far the analytical expression for the displacement in such a case differs from those already considered. Suppose then that, for values of x between $+a$ and $-a$, the initial displacement is c, and that for all other values of x it is zero.

Then $$y_0 = \frac{c}{\pi}\int_0^\infty \frac{\sin(x+a)\,m - \sin(x-a)\,m}{m}\,dm,$$

and $$y = \frac{c}{\pi}\int_0^\infty \frac{\sin(x+a)\,m - \sin(x-a)\,m}{m}\cos m^{\frac{1}{2}}g^{\frac{1}{2}}t\,dm.$$

The integral to whose value it is necessary to approximate in this case is

$$\int_0^\infty \frac{\sin m \cos m^{\frac{1}{2}} p^{\frac{1}{2}}}{m}\, dm,$$

where
$$p = \frac{gt^2}{x \pm a},$$

and it may be readily expressed in a series of ascending powers of p from which, for moderate values of p, the displacement may be calculated.

Thus, if $f(p)$ denote the integral,

$$f(p) = \int_0^\infty dv \int_0^\infty e^{-mv} \sin m \sum_0^\infty (-1)^n \frac{p^n}{2n!} m^n dm$$

$$= \frac{1}{2i} \int_0^\infty dv \sum_0^\infty (-1)^n \frac{n!}{2n!} p^n \left(\frac{1}{(v-i)^{n+1}} - \frac{1}{(v+i)^{n+1}} \right)$$

$$= \frac{\pi}{2} + \sum_1^\infty (-1)^{n+1} \frac{2n!}{4n+2!} p^{2n+1},$$

and
$$y = \frac{c}{\pi} \left[f\left(\frac{gt^2}{x+a}\right) - f\left(\frac{gt^2}{x-a}\right) \right].$$

As long as gt^2/x is small compared with x/a the above value of y is clearly of the order a/x, and as we may now anticipate that the greatest displacement at any point is of the order $(a/x)^{\frac{1}{2}}$, it is necessary to consider the value of $f(p)$ for large values of p for which purpose the series is of little use.

Writing v^2 for mp in the integral, we have

$$f(p) = 2 \int_0^\infty \frac{\sin v^2/p \cos v}{v}\, dv$$

$$= 2 \left[\frac{\sin v^2/p \sin v}{v} \right]_0^\infty - \frac{4}{p} \int_0^\infty \cos v^2/p \sin v\, dv + 2 \int_0^\infty \frac{\sin v^2/p \sin v}{v^2}\, dv.$$

The integrated term vanishes at the limits, and the last term

$$= 2 \int_p^\infty \frac{dp}{p^2} \int_0^\infty \cos v^2/p \sin v\, dv,$$

therefore
$$f(p) = \left[-\frac{4}{p} + 2 \int_p^\infty \frac{dp}{p^2} \right] \int_0^\infty \cos v^2/p \sin v\, dv.$$

Again, $\int_0^\infty \cos v^2/p \sin v \, dv = \frac{1}{2}\int_0^\infty \sin\left[\left(\frac{v}{p^{\frac{1}{2}}} + \frac{p^{\frac{1}{2}}}{2}\right)^2 - \frac{p}{4}\right] dv$

$$-\frac{1}{2}\int_0^\infty \sin\left[\left(\frac{v}{p^{\frac{1}{2}}} - \frac{p^{\frac{1}{2}}}{2}\right)^2 - \frac{p}{4}\right] dv$$

$$= \frac{1}{2}p^{\frac{1}{2}}\cos\frac{p}{4}\left[\int_{\frac{1}{2}p^{\frac{1}{2}}}^\infty \sin u^2 du - \int_{-\frac{1}{2}p^{\frac{1}{2}}}^\infty \sin u^2 du\right]$$

$$-\frac{1}{2}p^{\frac{1}{2}}\sin\frac{p}{4}\left[\int_{\frac{1}{2}p^{\frac{1}{2}}}^\infty \cos u^2 du - \int_{-\frac{1}{2}p^{\frac{1}{2}}}^\infty \cos u^2 du\right]$$

$$= \frac{1}{2}p^{\frac{1}{2}}\cos\frac{p}{4}\left[-\sqrt{\frac{\pi}{2}} + 2\int_{\frac{1}{2}p^{\frac{1}{2}}}^\infty \sin u^2 du\right]$$

$$-\frac{1}{2}p^{\frac{1}{2}}\sin\frac{p}{4}\left[-\sqrt{\frac{\pi}{2}} + 2\int_{\frac{1}{2}p}^\infty \cos u^2 du\right];$$

and, using the values of the integrals already given in a slightly different form,

$$= \frac{1}{2}p^{\frac{1}{2}}\cos\frac{p}{4}\left[-\sqrt{\frac{\pi}{2}} + \frac{\cos p/4}{(p/4)^{\frac{1}{2}}} + \frac{1}{2}\,\frac{\sin p/4}{(p/4)^{\frac{3}{2}}} - \frac{1\,.\,3}{2\,.\,2}\,\frac{\cos p/4}{(p/4)^{\frac{5}{2}}} - \dots\right]$$

$$-\frac{1}{2}p^{\frac{1}{2}}\sin\frac{p}{4}\left[-\sqrt{\frac{\pi}{2}} - \frac{\sin p/4}{(p/4)^{\frac{1}{2}}} + \frac{1}{2}\,\frac{\cos p/4}{(p/4)^{\frac{3}{2}}} + \frac{1\,.\,3}{2\,.\,2}\,\frac{\sin p/4}{(p/4)^{\frac{5}{2}}} - \dots\right]$$

$$= -\frac{\sqrt{\pi}}{2}p^{\frac{1}{2}}\cos\left(\frac{p}{4} + \frac{\pi}{4}\right) + 1 - \frac{1\,.\,3}{2\,.\,2}\left(\frac{4}{p}\right)^2 + \dots,$$

where, as before, the divergent series may be used as an approximation by stopping before the smallest term.

The leading part of

$$\int_p^\infty \frac{dp}{p^2}\int_0^\infty \cos v^2/p \sin v \, dv,$$

when p is great, will be

$$-\frac{1}{2}\sqrt{\pi}\int_p^\infty \frac{\cos(p+\pi)/4}{p^{\frac{3}{2}}}dp + \int_p^\infty \frac{dp}{p^2},$$

or

$$\frac{1}{p} + \text{terms of order } \frac{1}{p^{\frac{3}{2}}}, \text{ etc.};$$

hence $f(p) = 2\sqrt{\pi}\,\frac{1}{p^{\frac{1}{2}}}\cos\frac{p+\pi}{4} - \frac{2}{p} + \text{terms of order } \frac{1}{p^{\frac{3}{2}}}$, etc.

The resulting value of the displacement is

$$y = \frac{2c}{\sqrt{\pi}}\left[\left(\frac{x+a}{gt^2}\right)^{\frac{1}{2}}\cos\left(\frac{gt^2}{4(x+a)}+\frac{\pi}{4}\right)-\left(\frac{x-a}{gt^2}\right)^{\frac{1}{2}}\cos\left(\frac{gt^2}{4(x-a)}+\frac{\pi}{4}\right)\right]$$
$$-\frac{2c}{\pi}\left[\frac{x+a}{gt^2}-\frac{x-a}{gt^2}\right]$$
$$+\text{ etc.};$$

and when, as before, terms of the order a/x multiplied into those retained are neglected,

$$y = \frac{2c}{\sqrt{\pi}}\left(\frac{x}{gt^2}\right)^{\frac{1}{2}}\left[\cos\left(\frac{gt^2}{4(x+a)}+\frac{\pi}{4}\right)-\cos\left(\frac{gt^2}{4(x-a)}+\frac{\pi}{4}\right)\right.$$
$$\left.+\frac{a}{2x}\left\{\cos\left(\frac{gt^2}{4(x+a)}+\frac{\pi}{4}\right)+\cos\left(\frac{gt^2}{4(x-a)}+\frac{\pi}{4}\right)\right\}\right].$$

So long as gt^2/x, though considerable, is small compared with x/a, the two pairs of terms in the bracket are both of the order a/x. When, however, gt^2/x and x/a are of the same order of magnitude, the second pair of terms are of the order a/x compared with the first. Neglecting, therefore, the last pair, as continually before, and for reasons already given neglecting terms of the order gt^2a^2/x^3 in the arguments of the periodic terms, we have, finally,

$$y = \frac{4c}{\sqrt{\pi}}\left(\frac{x}{gt^2}\right)^{\frac{1}{2}}\sin\frac{gt^2a}{4x^2}\sin\left(\frac{gt^2}{4x}+\frac{\pi}{4}\right).$$

Here, again, $\sin\left(\frac{gt^2}{4x}+\frac{\pi}{4}\right)$ changes rapidly as compared with

$$\left(\frac{x}{gt^2}\right)^{\frac{1}{2}}\sin\frac{gt^2a}{4x^2},$$

and to determine the magnitude of the disturbance at any time the latter quantity need alone be considered. Its maximum values are given by

$$0 = \frac{1}{t^2}\sin\frac{gt^2a}{4x^2}-\frac{ga}{2x^2}\cos\frac{gt^2a}{4x^2},$$

or
$$\tan\theta = 2\theta,$$

where
$$\theta = \frac{gt^2a}{4x^2}.$$

The roots of this equation, omitting zero, form an ascending series

θ_1, θ_2, etc., such that θ_n approximates rapidly to $(2n-1)\,\tfrac{1}{2}\pi$ as n increases. The first two roots are approximately

$$\theta_1 = 1{\cdot}165,$$

$$\theta_2 = 4{\cdot}622.$$

To each root of the equation corresponds a maximum value of the amplitude for the particular value of x considered, that corresponding to θ_n being

$$\frac{2c}{\sqrt{\pi}}\left(\frac{a}{x}\right)^{\frac{1}{2}}\frac{\sin\theta_n}{\theta_n^{\frac{1}{2}}}.$$

Now $$\frac{\sin\theta_1}{\theta_1^{\frac{1}{2}}} = {\cdot}851,$$

and $$\frac{\sin\theta_2}{\theta_2^{\frac{1}{2}}} = -{\cdot}463,$$

and $$\frac{\sin\theta_n}{\theta_n^{\frac{1}{2}}} = (-1)^{n-1}\sqrt{\frac{2}{(2n-1)\,\pi}},\ \text{nearly},$$

so that the first of these successive maxima is the greatest in absolute magnitude, and they diminish continually.

The maximum corresponding to each root of the equation

$$\tan\theta - 2\theta = 0$$

is propagated with uniform velocity, and the magnitude of each maximum amplitude is, as before, directly proportional to the square root of the whole energy, and inversely as the square root of the distance from the original disturbance.

The intervals between successive maxima are periods of time comparable with that elapsing between the original disturbance and the first maximum.

A second feature distinguishing this case from those before, is in the much greater duration, *ceteris paribus*, of the sensible disturbance at a particular point.

If t_0 is the time elapsing between the original disturbance and the chief maximum at the point considered, $t_0/\log n$ is an approximate measure of the time that elapses in the former case before the amplitude diminishes to $1/n^{\text{th}}$ of its greatest value; while in the case now under consideration $(n-1)\,t_0$ is the corresponding quantity.

The integrals which will represent the form of the surface at any time, when the disturbance is caused by impulsive pressures applied over a limited extent of surface, are generally of slightly different form to those arising from the cause considered in this paper; but similar methods may be used to approximate to their values. The forms obtained for the surface and the general laws which the motion follows are, as is to be expected, precisely similar to the results obtained here.

On a certain Atomic Hypothesis. *By* Prof. KARL PEARSON.

[*Read Nov. 8th*, 1888.]

1. In a paper written in 1883, and read before the Cambridge Philosophical Society in 1885 (*Camb. Phil. Trans.*, XIV., Part II., pp. 71—120), I supposed, as a first approximation, that the ultimate constituents of matter might be treated as spherical bodies capable of surface vibration and pulsation. I then endeavoured to show that the free and forced vibrations of such atoms were sufficient to produce in a fluid ether many of the phenomena of chemical affinity, cohesion, and gravitation. The assumptions really made were:

(i.) The pressures produced on the surface of one atom by the vibrations of another are, at any rate to a first approximation, identical with those which would be produced were the ether a perfect fluid.

(ii.) The ultimate atom may, to a first approximation, be treated as a vibrating sphere.

In the course of my work, I did not pre-suppose any special internal structure for the spherical atom, beyond a capacity for absorbing energy in the form of surface vibrations. I also gave some reasons for the belief that a simple pulsation must play a more important part than polar vibrations within the sphere of inter-atomic influence. Further, the more complex molecule, which forms the basis of the

wider range of physical phenomena associated with certain bodies, was, I held, to be sought in a combination of the simpler spherical atoms rather than in a more complex atomic mechanism.

Sir William Thomson, in his Baltimore "Lectures on Molecular Dynamics," 1884, has treated of a molecule of considerable mechanical complexity, which fulfils to a very great extent the optical functions which we must suppose inherent in the basis of matter. That the ultimate atom does not consist of a material core surrounded by a number of material spherical shells, linked together by elastic springs, is, of course, obvious; but that the dynamical equations arising from this mechanical system may closely resemble those of the true atom, is the inherent merit of the discovery. At the same time, this mechanical system of Thomson's does not admit of separation into simpler elements, and in fact suggests no method whereby we may build up from the simplest chemical substances those having far more complex optical and other physical properties. In the paper referred to, I have endeavoured to show that groups of simple pulsating atoms may, to a considerable extent, fulfil the more complex physical and chemical conditions observed in bodies whose molecular structure is obviously of an intricate nature.

I had intended some years ago to point out further chemical and optical results which flow from the pulsating spherical atom; but the pressure of other work, as well as the limited interest which any artificial atomic system can claim, were sufficient to deter me. The recent publication of Prof. Dr. F. Lindemann's memoir* on Thomson's atom, together with my ever growing conviction that the mere translational and vibrational motions of the atoms in the ether itself are sufficient to explain all so-called chemical and physical forces, have induced me to attempt the explanation of sundry further chemical and physical phenomena on the basis of the pulsating atom.

The paper of Lindemann to which I have referred consists of two parts. In Part I., pp. 2—23, he gives an account of Thomson's mechanism, and, following Thomson fairly closely, he deduces explanations of a number of optical phenomena. In Part II., pp. 23—51, he deals in somewhat general terms with the magnetic and electrical results which flow from the hypotheses:

(i.) That the ether behaves with regard to the relatively long motions of the molecule as a perfect fluid.

* "Über Molekularphysik. Versuch einer einheitlichen dynamischen Behandlung der physikalischen und chemischen Kräfte," *Schriften der physikalisch-ökonomischen Gesellschaft zu Königsberg*, XXIX., Jahrgang 1888.

(ii.) That the molecule is a "Thomson's molecule."

(iii.) That the wave-lengths of the ether waves which excite the physical phenomena discussed are no longer immensely greater than the molecular diameter, but that conversely the molecular diameter is great as compared with these wave-lengths.

The first hypothesis here is also the first hypothesis of the pulsating atomic theory. The second hypothesis seems to be unnecessary. Lindemann's explanations, which are occasionally in rather general wording, could, I think, be applied to a considerable range of atomic mechanisms; in particular to an atom in a fluid ether capable of receiving and acting as a vehicle for vibrational energy drawn from the ether. The pulsating spherical atom is here, I think, equally efficient.

In the paper in the *Camb. Phil. Trans.*, no assumption was made as to the relative magnitudes of wave-length and atomic diameter; there is, however, nothing in my theory to hinder such an assumption being made, should it appear advantageous for the explanation of any natural facts.

In the following pages, I propose to discuss various physical phenomena on the basis of the vibrating spherical atom. In the course of my investigation, I shall refer to the pages of Lindemann's offprint where he has dealt in a similar manner with Thomson's atom.

I. *Dispersion of Light.*

2. Consider a molecule composed of n different pulsating spherical atoms. Then the typical equation for the free vibrations of the molecule with the notation of the *Camb. Phil. Trans.* paper, § 18, p. 89, is

$$(\lambda_r + 4\pi a_r^3)\,\ddot{\phi}_r + r_r\phi_r + \Sigma\, Q_{rs}\ddot{\phi}_s = 0 \quad \text{.................}(i.).$$

The equation for the vibration of the ether, if ρ be its density, and k its elastic constant, is

$$\rho\,\frac{d^2y}{dt^2} = k\,\frac{d^2y}{dx^2} \quad \text{..............................}(ii.),$$

y being the ether-shift, and x the direction of wave-propagation.

The question now arises: What alterations must be made in equations (i.) and (ii.), when the ether-wave is influenced by the presence of the molecule in its path? Sir W. Thomson supposes a final elastic spring to act between the last shell of his mechanism and the ether,

and consequently introduces a term of the form $c(y'-y)$ into equation (ii.), where y' is the shift of the centre of the last shell of his molecule. We are unable to introduce a term of like form, because the outer surface of our molecule is built up of n spherical surfaces of different radii, and their exact effect on equation (ii.) would not be easy to determine. But I think we may legitimately argue in somewhat the following manner. The effect of the ether-wave will only change to a limited extent the form of the molecular vibration functions ϕ_r, ϕ_r. The force acting on the ether, owing to the vibrations excited in the molecule, must have the same harmonics as these slightly changed functions, and its effect on equation (ii.) may be represented by a term of the form

$$c_1\phi_1+c_2\phi_2+\ldots+c_r\phi_r+c_s\phi_s+\ldots = \Sigma c_s\phi_s.$$

Equation (ii.) thus becomes

$$\rho\frac{d^2y}{dt^2} = k\frac{d^2y}{dx^2}+\Sigma c_s\phi_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)}.$$

This supposes that for a definite time only one wave of light reaches the molecule, or that its dimensions are small as compared with the wave-length.

Conversely, the effect of the ether on the molecular equations may be supposed to introduce a force into the normal equations proportional to the ether-shift y. Thus the normal equations for the vibrations forced on the molecule take the form

$$(\lambda_r+4\pi a_r^2)\,\ddot{\phi}_r+\tau_r\phi_r+\Sigma\,Q_{rs}\ddot{\phi}_s = b_r y\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv.)}.$$

We may, however, bring our equations nearer to the form of those of Thomson and Lindemann, by adding on the right-hand side of equations (iii.) and (iv.) respectively fy and $\Sigma e_{rs}\phi_s$, where f and e_{rs} are certain constants, they then become

$$\rho\frac{d^2y}{dt^2} = k\frac{d^2y}{dt^2}+fy+\Sigma c_s\phi_s\text{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v.)},$$

$$(\lambda_r+4\pi a_r^2)\,\ddot{\phi}_r+\tau_r\phi_r+\Sigma\,Q_{rs}\ddot{\phi}_s = b_r y+\Sigma e_{rs}\phi_s\ldots\ldots\ldots\text{(vi.)}.$$

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* This equation seems based on a fair assumption, if we suppose the pulsation of the s^{th} atom of any one molecule to be in the same phase as the corresponding atom in any other molecule in the face of the wave of light at the time it impinges on the body. This is probable, if we suppose it is the wave itself which excites the atoms (as Thomson in the case of the shell-atoms); on the other hand, if we suppose the atoms to be already pulsating, and only their amplitudes and periods to be more or less affected by the incident wave, we are thrown back on the hypothesis for which there appear to be other grounds, that corresponding atoms will always be pulsating in like phase; see the *Camb. Phil. Trans.* paper, pp. 107, 108.

Equations (v.) and (vi.) allow for a certain amount of viscous or slipping action between the molecule and the ether, and it seems as legitimate to introduce it off-hand, as to deduce these equations by placing a "spring" between the molecule and the ether. The introduction of a fresh series of unknown constants is, of course, a disadvantage, but the series may always be omitted if it be not found necessary for the explanation of any physical phenomena. The term fy will make a difference at least in the *form* of the results. Thomson has in (v.) a single term proportional to the difference between the ether-shift and the central shift of the outermost spherical shell of the molecule. In our case, ϕ_s is the surface shift of the s^{th} atom of the molecule, and thus $\Sigma c_s \phi_s$ is a function of the surface shifts of all the atoms of the molecule; it is therefore possible to regard $fy + \Sigma c_r \phi_r$ as the mean difference of the ether and the molecular surface shift, and so obtain an equation which has at least the same amount of justification as Thomson's, based on the final ether "spring."

With regard to the quantity $\Sigma e_{rs} \phi_s$, I may remark that the principal, if not the sole, term of this in the r^{th} typical equation will in all probability be $e_{rr} \phi_r$. But, in this case, e_{rr} will simply alter the value of τ_r, that is, will alter apparently the potential energy capacity of the r^{th} atom, and so of the entire molecule. Thus, *one* effect of placing the molecule in the midst of an ether-wave would be to slightly alter the periods of molecular vibration, or to shift its spectrum lines. If such shifting actually takes place, we should have a means of determining the sign of e_{rr}.

3. Let $y = A \cos \frac{2\pi}{l}(vt - x)$ give the wave of light; let $T = l/v$ be its period, l its length, v its speed, and $\mu = 1/v$ the refractive index. Further, let $m_r = \lambda_r + 4\pi a_r^3$ for brevity. Then, substituting in (v.) and (vi.) $\phi_s = B_s y$, we have

$$-\rho \frac{4\pi^2}{T^2} y = -k \frac{4\pi^2}{T^2 v^2} y + fy + \Sigma c_s B_s y,$$

$$-m_r \frac{4\pi^2}{T^2} B_r y + \tau_r B_r y - \frac{4\pi^2}{T^2} \Sigma Q_{rs} B_s y = b_r y + \Sigma e_{rs} B_s y.$$

Dividing by y and rearranging terms, etc., we have

$$\sum_1^n c_s B_s + f + \rho \frac{4\pi^2}{T^2} - k \frac{4\pi^2}{T^2} \mu^2 = 0 \quad \text{.................(vii.)},$$

$$\sum_{1 \text{ to } r-1}^{r+1 \text{ to } n} B_s \left(e_{rs} + \frac{4\pi^2}{T^2} Q_{rs} \right) + B_r \left(e_{rr} - \tau_r + m_r \frac{4\pi^2}{T^2} \right) + b_r = 0 \quad \text{... (viii.)}.$$

Hence, eliminating by a determinant, we have the following equation between the refractive index μ and the period T:

$$\left|\begin{array}{ccccc} c_1, & c_2, & c_3, \;\ldots\ldots & c_n, & f+\rho\frac{4\pi^2}{T^2}-k\frac{4\pi^2}{T^2} \\ m_1\frac{4\pi^2}{T^2}+e_{11}-\tau_1, & e_{12}+\frac{4\pi^2}{T^2}Q_{12}, & e_{13}+\frac{4\pi^2}{T^2}Q_{13}, \;\ldots\ldots & e_{1n}+\frac{4\pi^2}{T^2}Q_{1n}, & b_1 \\ e_{21}+\frac{4\pi^2}{T^2}Q_{12}, & m_2\frac{4\pi^2}{T^2}+e_{22}-\tau_2, & e_{23}+\frac{4\pi^2}{T^2}Q_{23}, \;\ldots\ldots & e_{2n}+\frac{4\pi^2}{T^2}Q_{2n}, & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ e_{n1}+\frac{4\pi^2}{T^2}Q_{n1}, & e_{n2}+\frac{4\pi^2}{T^2}Q_{n2}, & \ldots\ldots\ldots\ldots & m_n\frac{4\pi^2}{T^2}+e_{nn}-\tau_n, & b_n \end{array}\right|$$

$$= 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i}$$

There appears no reason why the value of μ, as given by the determinant (ix.), should not satisfy the physical conditions as well as the formula given by Lindemann (pp. 4—8). It so far has the advantage that it allows us to see how any physical cause (*e.g.* pressure) which alters the inter-atomic distance, and therefore Q_{rs} $(= 4\pi a_r^2 a_s^2/\gamma_{rs}$, where γ_{rs} = inter-atomic distance) affects the refractive index. It further admits of chemical changes taking place in the molecule, *i.e.*, atoms being removed, and indicates generally the nature of the resultant changes in μ^2.

4. It may, of course, be necessary for some physical purposes to deal with the formula (ix.) in its most general form, especially if we are to explain the more complex optical phenomena. At the same time some light will be thrown on the general bearing of our theory, if for the present we limit the character of the hypotheses on which we have based equations (v.) and (vi$_r$). Let us then first suppose that e_{rs} is negligible when r and s are different. In the next place, for brevity, we may put $e_{rr}-\tau_r = -\tau'_r$, and, just as $\nu_r^2 = \tau_r/m_r$, we may write $\nu_r'^2 = \tau'_r/m_r$, so that $2\pi/\nu'_r$ = the period of the r^{th} atom supposed freed from the molecule, and influenced only by ether vibrations; we may write this p_r. Thus equation (viii.) becomes

$$\sum_{1\text{ to }r}^{r+1\text{ to }n}{}_{1}\; B_s\frac{4\pi^2}{T^2}Q_{rs}+B_r\tau'_r p_r^2\left(\frac{1}{T^2}-\frac{1}{p_r^2}\right)+b_r = 0 \;\ldots\ldots\ldots\ldots(\text{x.}).$$

Neglecting as a first approximation the terms involving Q_{rs},

we have

$$B_r = \frac{b_r T^2}{\tau_r' (T^2 - p_r^2)} \quad \text{.............................(xi.)},$$

which is allowable so long as T is not nearly equal to the "reduced free atomic period" p_r.

Substituting this value of B in the terms under the summation, we have

$$B_r \frac{\tau_r' (p_r^2 - T^2)}{T^2} + b_r + \sum_{1 \text{ to } r-1}^{r+1 \text{ to } n} \frac{4\pi^2 b_s Q_{rs}}{\tau_s' (T^2 - p_s^2)} = 0,$$

or,
$$B_r = \frac{b_r T^2}{\tau_r' (T^2 - p_r^2)} + \frac{4\pi^2 T^2}{\tau_r' (T^2 - p_r^2)} \sum_{1 \text{ to } r-1}^{r+1 \text{ to } n} \frac{b_s Q_{rs}}{\tau_s' (T^2 - p_s^2)} \quad \text{.........(xii.)}.$$

Substituting in equation (vii.), we find for μ^2

$$\mu^2 = \frac{\rho}{k} + \frac{f}{4\pi^2 k} T^2 + \frac{T^4}{4\pi^2 k} \sum_1^n \left\{ \frac{c_s b_s}{\tau_s' (T^2 - p_s^2)} + \frac{4\pi^2 c_s}{\tau_s' (T^2 - p_s^2)} \sum_{1 \text{ to } s-1}^{s+1 \text{ to } n} \frac{b_r Q_{rs}}{\tau_r' (T^2 - p_r^2)} \right\}$$
$$\text{..................(xiii.)}.$$

This formula in itself is very general, and seems sufficient to explain most of the phenomena of dispersion. I proceed to consider special cases.

(*a*) A mon-atomic molecule, or a molecule in which, for one atom only, the term $\frac{c_s b_s}{\tau_s' (T^2 - p_s^2)}$ is not very small. Here we have

$$\mu^2 = \frac{\rho}{k} \left\{ 1 + \frac{f}{4\pi^2 \rho} T^2 + \frac{cb}{4\pi^2 \tau' \rho} \frac{T^4}{T^2 - p^2} \right\} \quad \text{............(xiv.)}.$$

If T is *nearly proportional to the wave-length* l, this equation may be written in the form

$$\mu^2 = C_1 + C_2 l^2 + C_3 \frac{l^4}{l^2 - l_0^2}.$$

In this form it coincides with that deduced from Helmholtz's theory, for which there is sufficient experimental verification. It agrees with the result of Thomson's theory, given on p. 6 of Lindemann's essay.

(*b*) Suppose we may neglect inter-atomic influence in the molecule. Then (xiii.) becomes

$$\mu^2 = \frac{\rho}{k} \left\{ 1 + \frac{f}{4\pi^2 \rho} T^2 + \frac{1}{4\pi^2 \rho} \sum_1^n \frac{T^4 c_s b_s}{\tau_s' (T^2 - p_s^2)} \right\} \quad \text{............(xv.)}.$$

This result is identical, so far as the expression of μ^2 in terms of T^2,

and p_e^2 with the fundamental result of Thomson [Lindemann, Equation (8), p. 4]. *Hence all the results drawn by Thomson and Lindemann concerning refraction and dispersion, so far as they are based on this equation, especially the explanation of anomalous dispersion, hold for the molecule built-up of pulsating spherical atoms, if to a first approximation we neglect the second summation of equation* (xiii.), *depending on inter-atomic influence in the molecule.**

The existence of these inter-atomic terms, however, is useful in explaining other phenomena, and indicates how a complex molecule can be built-up from the simpler atoms. It is obvious that Thomson's mechanism is only one out of many which will furnish his fundamental equation, and, in choosing any one of these to represent the molecule, we must be guided by the width of the range of phenomena which the individual mechanism will explain, as well as its inherent physical possibility. A pulsating atom as basis of the most complex molecule seems *per se* more probable than an indivisible spring mechanism, if it is capable of giving as wide a range of results.

I shall not discuss further at present the application of equation (xiv.) to the phenomena of dispersion and reflection. The reader will find this considered at length by Thomson in his *Lectures*, and Lindemann in his memoir (see especially, for a modification of Thomson's view, p. 9).

(*c*) *Case of a di-atomic molecule.*

Writing $\beta = f/4\pi^2\rho$, $\beta_1' = c_1 b_1/4\pi^2\rho\tau_1'$, $\beta_2' = c_2 b_2/4\pi^2\rho\tau_2'$,

and
$$\beta'' = \frac{c_1 b_2 + c_2 b_1}{\tau_1'\tau_2'\rho},$$

we easily find

$$\mu^2 = \frac{\rho}{k}\left\{1+\beta T^2 + T^4\left(\frac{\beta_1'}{T^2-p_1^2}+\frac{\beta_2'}{T^2-p_2^2}\right)+\frac{T^4\beta''Q_{12}}{(T^2-p_1^2)(T^2-p_2^2)}\right. \dots \text{(xvi.)}.$$

This, in conjunction with $T = \mu l$, gives the relation between the refractive index and the wave-length. By expanding in powers of T/p_1, T/p_2, or of p_2/T, p_1/T, or again in powers of T/p_1 and p_2/T, we obtain formulæ corresponding to the usual expressions for the refractive index in terms of the wave-length, and explaining to a great extent the phenomena of anomalous dispersion. It would appear that in most cases the coefficient of T^2 in the expansions is very small.

* The exact equivalence of these equations with the Thomson-Lindemann equations depends on the assumption that there really exists a term in our equations like fy. It may be doubted whether the term in μ involving f is of any physical importance.

The last term of the above expression for μ^2 gives the part of the refractive index depending on inter-atomic action in the molecule, and this, under ordinary circumstances of pressure, temperature, etc., is probably small.

The result for an n-atomic molecule may be written down in a similar form, *i.e.*,

$$\mu^2 = \frac{\rho}{k}\Big[1+\beta T^2+T^4\Big\{\frac{\beta_1'}{T^2-p}+\frac{\beta_2'}{T^2-p_2^2}+\frac{\beta_3'}{T^2-p_3^2}+\dots\Big\}$$
$$+\frac{T^4\beta_{12}''Q_{12}}{(T^2-p_1^2)(T^2-p_2^2)}+\frac{T^4\beta_{13}''Q_{13}}{(T^2-p_1^2)(T^2-p_3^2)}+\frac{T^4\beta_{23}''Q_{23}}{(T^2-p_2^2)(T^2-p_3^2)}+\dots\Big] \quad \text{(xvii.)}$$

(*d*) *Case of an n-equi-atomic molecule.*

The case of all the atoms in a molecule being equal is interesting, even if such molecules are infrequent, for it seems probable that the atomic periods $p_1, p_2 \dots$ may often approach sufficiently closely to be treated as equal. It should be noticed, however, that this equality really involves more than equal atomicity. If the atoms are equal, their radii a, their kinetic and potential energy capacities λ and τ are the same. This involves the equality of all the m's $(= \lambda+4\pi a^3)$, but not necessarily that of the p's $(= 2\pi/\nu'$ where $\nu'^2 = \tau'/m)$. This latter involves the equality of the τ''s $(= \tau-e_{rr})$, or the wave in the ether *must affect the potential energy capacity of all the equal atoms in like manner.* This may require the wave to strike the molecule in a manner equi-symmetrical to all its equal atoms. We can thus understand how it is possible for the refractive index to be given by a different formula for different directions. Thus a ray of light passing through a body might be normally refracted in one direction, and have anomalous dispersion in a second. At the same time in many cases the e_{rr}'s may be so nearly equal, that we are justified in treating the p's as such.

On this supposition, we obtain for an n-equi-atomic molecule, the formula

$$\mu^2 = \frac{\rho}{k}\Big[1+C_1T^2+\frac{C_2T^4}{T^2-p^2}+\frac{C_3T^4}{(T^2-p^2)^2}\Big] \quad \text{(xviii.)},$$

where C_3 is a function of the inverse inter-atomic distances.

This formula (except for the probably small influence of the inter-atomic action) closely resembles (xiv.), and we see that our theory leads to Helmholtz's result, without any necessity for reducing our molecule to an unusually simple form. Indeed, in Thomson's

mechanism, equality of periods, although it is possible, yet involves rather complex relations between his springs and shells; the probability of the existence of these relations is hardly so great as the probability of the existence of equal atoms.

5. I shall now consider the results which arise when we suppose the ether to be disturbed by a wave of the same or nearly the same period as the " reduced free atomic period " p_r.

The result of this is that the term involving B_r in the r^{th} equation vanishes, and we are not able to neglect the sum term of equation (viii.), § 3, to a first approximation. If we assume, as before, that e_{rs} is very small or zero, when r and s differ, we see that B_s will have for denominator a very small term of form $e_{rs}+\frac{4\pi^2}{T^2}Q_{rs}$, and B_r a term involving the product of such quantities. Thus the amplitude of atomic pulsation will be very great. The complete results are easily written down in determinant form from equations (vii.) and (viii.), omitting in one of the latter type the second term. In the case when all the atoms are equal, all the second terms of the equations of this type must be omitted; the result gives amplitudes of only the inverse, and not the inverse product of small quantities. In other words: *The individual atoms of an equi-atomic molecule will be less excited by a wave of their critical period, than those of a molecule built-up of unequal atoms.* This suggests that molecules built-up of equal atoms may be harder to dis-associate by light or heat than those formed of unequal atoms.

The above will be best exemplified by taking the case of a di-atomic molecule, in the first place with unequal periods p_1 and p_2; let the former be also the period of the ether wave. Then, from equation (viii.), we have

$$B_2\left(e_{12}+\frac{4\pi^2}{T^2}Q_{12}\right)+b_1=0,$$

$$B_1\left(e_{21}+\frac{4\pi^2}{T^2}Q_{21}\right)+B_2r_2'\left(\frac{p_2^2}{T^2}-1\right)+b_2=0;$$

whence we find $$B_2=-b_1\bigg/\left(e_{12}+\frac{4\pi^2}{T^2}Q_{12}\right),$$

$$B_1=-b_2\bigg/\left(e_{21}+\frac{4\pi^2}{T^2}Q_{21}\right)-\frac{r_2'b_1\,(T^2-p_2^2)}{T^2\left(e_{21}+\frac{4\pi^2}{T^2}Q_{21}\right)\left(e_{12}+\frac{4\pi^2}{T^2}Q_{12}\right)}.$$

In the case we have supposed, when e_{21}, e_{12} are absolutely

negligible, we have

$$B_2 = -\frac{b_1 T^2}{4\pi^2 Q_{12}},$$

$$B_1 = -\frac{b_2 T^2}{4\pi^2 Q_{12}} - \frac{b_1 r_2' T^2 (T^2 - p_2^2)}{16\pi^4 Q_{12}^2}.$$

Thus the amplitudes, depending on the inverse of Q_{12}, are very great, especially B_1, which involves a term containing $1/Q_{12}^2$. The same general results arise if e_{21} and e_{12} are of the same order of magnitude as Q_{12}. Thus the finiteness of the values we have obtained for B_1, B_2, depends on the existence of the inter-atomic action. In the one case when the two atoms are equal, $T^2 = p_2^2$ also, B_1 and B_2 will be of the same order of magnitude, but otherwise the first atom will be considerably more excited than its fellow or than either of a pair of twin atoms; *i.e.*, an unequal di-atomic molecule will be, as a rule, more easily disassociated than a twin-atomic molecule.

Equation (vii.) gives for the refractive index

$$\mu^2 = \frac{\rho}{k}\left\{1 + \frac{f}{4\pi^2\rho}T^2 - \frac{c_1 b_2 T^4}{16\pi^4\rho\, Q_{12}} - \frac{c_2 b_1 T^4}{16\pi^4\rho\, Q_{12}} - \frac{b_1 c_1 r_2' T^4 (T^2 - p_2^2)}{64\pi^6\rho\, Q_{12}^2}\right\} \qquad \text{(xiv.)}.$$

In all probability the last term as involving $1/Q_{12}^2$ is the all-important term, and, if endowed with a negative sign, it may give μ^2 = a negative quantity, and correspond to complete reflection, if there be not accompanying incandescence.

The effect of amplitudes of pulsation-vibration like B_1 and B_2, as given above, in breaking up a molecule, has been dealt with in the *Camb. Phil. Trans.* paper. It is obvious that these forced pulsations may produce repulsive forces exceeding those of "chemical affinity" (Part I., p. 102), and so waves of light or heat may produce chemical disassociation or physical dissolution.

It may be noted that, other things being equal, the ether waves excite an atom most when their period T is greatest; in other words, the ultra-red or "heat" rays should be, as a rule, most active in the work of disassociation. Obviously, the absorption and radiation of rays of the same wave-length by a given material is intimately related to the above investigation. We may apparently add the principle: *That a molecule will be most easily chemically disassociated by waves of the same period as the bright lines of its own spectrum*,—a result which would probably follow from the majority of dynamical systems, which might be taken as molecular basis.

The so-called curve of maximum chemical action of the various wave-lengths of a ray of light probably only measures the nearness of the wave period to a "reduced free atomic period," and has relation only to the individual material which is used as a light receiver. Its form might well be suggested by an equation similar to that connecting B_1 and T^2 above.

6. The discussion which Lindemann gives on p. 5 as to the relations of the molecule of Thomson to the kinetic theory of gases, applies equally well to the pulsating spherical atom. Indeed, it is of such a general nature that it would apply to most conceivable molecular mechanisms, in which there may be a limit to the absorption of internal energy. So soon as the vibrational energy has reached its maximum, if more energy be given to the molecule, then either chemical dissociation or translational energy—one form of heat—must make its appearance.

7. *Double Refraction* is dealt with by Lindemann in § 6 (pp. 12—14). His theory involves a different set of springs between the shells for each direction, and the assumption that a certain function of the spring constants is given for a direction (α, β, γ) by the formula

$$\left(\frac{c_1}{c_1+c_2}\right)^2 = C_1 \cos^2 \alpha + C_2 \cos^2 \beta + C_3 \cos^2 \gamma \quad \text{(p. 13, equation 22).}$$

This great increase of the complexity of the molecule is in itself unsatisfactory; even if we suppose the molecule to be an elastic sphere whose aeolotropy is arranged in some ellipsoidal fashion as suggested above, we are still at a loss to understand how the chemical decomposition of molecules can take place. The atomic theory, which starts from the pulsating sphere, builds up its molecule from the simple atoms; and it follows that, when the body does not possess confused crystallisation, *i.e.*, when the molecules are not turned in every conceivable direction, but have the same orientation, then the *direction* of the ether wave will be of importance. In other words, a group of n spheres has not generally the same aspect for every direction, and thus the constants f, c, b, and e, of equations (v.) and (vi.), will be functions of the direction of the ether wave.

Since Lindemann only starts from an equation similar to our equation (xvii.), and then, by making various assumptions as to the relations between his constants, reaches an explanation of double refraction, there is nothing to hinder our following in his footsteps, and deducing for our theory an explanation of at least equal validity. At the same time it may well be questioned whether such a process

carries much weight of conviction with it, and whether the discussion of the subject is not better postponed until the mathematical theory of the effect of a group of spheres on an ether wave has been more thoroughly studied than at present.

8. On pp. 14—22, Lindemann deals with various problems concerning the spectra of compounds, and the chemical effects of light and heat. These points have been fully dealt with in my first paper from an entirely different standpoint. Such general discussions as the pages referred to contain, will hardly determine the value of a theory—that must finally depend on the actual test of figures. It may be noted that the Thomsonian atoms, or molecules according to Lindemann, remain associated without the aid of any inter-atomic force, because association is a position in which they have less total vibrational energy than when they are independent; thus work must be expended in separating them.

I may note a point in which the Thomson-Lindemann theory differs essentially from that of the pulsating sphere. In the former the spectral lines of a molecule built up out of two like atoms are the same as for the free atom; in the latter (*Camb. Phil. Trans.*, p. 88, § 17) they differ. The former theory also has no explanation why the molecules of a gas may possibly be built up of two equal atoms (Lindemann, p. 20).

9. *Fluorescence.*—Suppose a wave of light, of nearly the same period as a "reduced period" of any atom of a molecule, to be incident upon the molecule, then it excites not only the normal function of the reduced period but the other normal vibrations as well. If, then, the capacity for absorbing energy corresponding to the normal function of the same period as the light wave be considerable, but that corresponding to a second reduced period, not that of the light wave, be small, then the body may give off light of the latter period, and not that of the incident light.

For example, in Art. 5, we have seen that the amplitudes B_1 and B_2 of the first and second atoms are both great. It is possible, however, that the first atom has a greater capacity for absorbing energy than the second, which will accordingly, when the source of light is removed, appear, owing to the magnitude of the pulsations forced upon it by the incident light, to give off light of its own (molecularly modified) period. For fluorescence to take place, then, it is necessary that the incident light should be nearly of the same period as one of the critical periods of the molecule (*cf.* Lindemann, § 11, p. 22).

II. *On Statical Electricity.*

10. Suppose it possible to set the ether in the neighbourhood of a spherical atom in motion with a period T, and to maintain this motion. Then the atom will also pulsate with the like period T, and according to equation (xi.), Art. 4, its amplitude will be of the form

$$B_r y = \frac{b_r T^2 y}{\tau_r' (T^2 - p_r^2)},$$

provided T is not equal to the critical period p_r.

If there be a second atom round which the ether also vibrates with period T, we have for its motion

$$B_s y = \frac{b_s T^2 y}{\tau_s' (T^2 - p_s^2)}.$$

Now, when two atoms pulsate with beats given by ϕ_0, ϕ_0', there arises a term in the kinetic energy of the system which may be written

$$\frac{4\pi a_r^2 a_s^2}{\gamma} \rho \dot{\phi}_0 \dot{\phi}_0' \quad (\textit{Camb. Phil. Trans.}, \text{ p. } 82).$$

This corresponds to an apparent term of the same form in the force-function of the two atoms. Thus, if $\dot{\phi}_0$ and $\dot{\phi}_0'$ are both positive, the atoms will attract each other with a force varying as the inverse square; and, if the atoms can approach each other, the result is an *increase* in the kinetic energy of the whole system. This supposes the vibrations $\dot{\phi}_0$, $\dot{\phi}_0'$ to be maintained (as in Bjerknes' Paris experiments) by some cause external to the ether. Now, let us suppose that it is the vibrations already existing in the ether itself, which excite pulsations in the atoms; they then absorb kinetic energy from the ether, and this must indicate the loss of just as much kinetic energy in the ether as was previously gained. In other words, the term in the apparent force-function, as well as the force, must be reversed, and we have for the two atoms, when their pulsations are produced by ether waves, the force-function*

$$-\frac{4\pi a_r^2 a_s^2}{\gamma} \rho \dot{\phi}_0 \dot{\phi}_0'.$$

* The principle involved here is of the following kind. Suppose energy to be communicated to the atoms by some means other than the ether (thus: by internal clock-work, or by an external air-pump communicating with the pulsating body by a tube as in Bjerknes' Paris experiments), then the tendency of the atoms is to place themselves in a position involving a greater potential energy of the system. For example, if there is a limit to the amplitude of the pulsation, or to the amount of

In our particular case, this becomes on substituting for $\dot{\phi}_0$ and $\dot{\phi}_0'$, if

$$dy/dt = 2\pi/T.y_1,$$

$$U = -\frac{4\pi a_r^2 a_s^2 \rho}{\gamma} \frac{b_r b_s}{r_r' r_s'} \frac{4\pi^2 T^2 y_1^2}{(T^2-p_r^2)(T^2-p_s^2)} \text{...........(xxiii.)}.$$

This leads to the following results :—

(i.) There is an attractive force varying as the inverse square when T lies between p_r and p_s.

(ii.) There is a repulsive force varying as the inverse square when T does not lie between p_r and p_s.

It follows :—

(iii.) That, when two pulsating spherical atoms are situated in ether vibrating with a given period, they will attract or repel each other with a force varying as the inverse square, according as the period of the ether lies or does not lie between the free periods of the atoms.

The law becomes more complex in the case of molecules. If $a_1, a_2, a_3 \ldots$ be the radii, $c_1, c_2, c_3 \ldots$ the constants (where $c_r = b_r/r_r'$), $p_1, p_2, p_3 \ldots$ the periods for the atoms of one molecule, and the same quantities with dashed letters be used for the second molecule, then we have in the force-function the term

$$-\frac{16\pi^3 \rho T^2 y_1^2}{\gamma}\left\{\frac{a_1^2 c_1}{T^2-p_1^2} + \frac{a_2^2 c_2}{T^2-p_2^2} + \frac{a_3^2 c_3}{T^2-p_3^2} + \ldots\right\}$$
$$\times\left\{\frac{a_1'^2 c_1'}{T^2-p_1'^2} + \frac{a_2'^2 c_2'}{T^2-p_2'^2} + \frac{a_3'^2 c_3'}{T^2-p_3'^2} + \ldots\right\},$$

where γ is the intermolecular distance.

Obviously, the conditions for the force being attractive or repulsive now become more complex, and depend upon the factors in brackets having the like or unlike signs. If $\sigma_1, \sigma_2, \sigma_3$ be the atomic surface

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surface energy in the atom, the additional energy put into the atom after this limit is reached must appear as potential energy of atomic position (*cf.* Lindemann, pp. 26-7). Hence the apparent force between two atoms will be in the direction which causes their potential energy of position to increase. On the other hand, if the pulsations are caused by disturbance in the ether itself, the atoms must absorb energy from the ether, and their tendency will be to place themselves in a position of less potential energy; the apparent force will thus be reversed.

areas for the one, and σ_1', σ_2', σ_3' for the other molecule, we have, for U,

$$U = -\frac{\pi\rho T^2 y_1^2}{\gamma}\left\{\frac{\sigma_1 c_1}{T^2-p_1^2}+\frac{\sigma_2 c_2}{T^2-p_2^2}+\frac{\sigma_3 c_3}{T^2-p_3^2}+\dots\right\}$$
$$\times\left\{\frac{\sigma_1' c_1'}{T^2-p_1'^2}+\frac{\sigma_2' c_2'}{T^2-p_2'^2}+\frac{\sigma_3' c_3'}{T^2-p_3'^2}+\dots\right\}\dots\text{(xxiv.)}.$$

We may note three special cases:—

(i.) T very much greater than all the free periods of the atoms: we have

$$U = -\frac{\pi\rho y_1^2}{\gamma}\frac{1}{T^2}\left\{\sigma_1 c_1+\sigma_2 c_2+\sigma_3 c_3+\dots\right\}\left\{\sigma_1' c_1'+\sigma_2' c_2'+\sigma_3' c_3'+\dots\right\}$$

(ii.) T very much less than all the free periods of the atoms: we have

$$U = -\frac{\pi\rho y_1^2 T^2}{\gamma}\left\{\frac{\sigma_1 c_1}{p_1^2}+\frac{\sigma_2 c_2}{p_2^2}+\frac{\sigma_3 c_3}{p_3^2}+\dots\right\}\left\{\frac{\sigma_1' c_1'}{p_1'^2}+\frac{\sigma_2' c_2'}{p_2'^2}+\frac{\sigma_3'\sigma_3'}{p_3'^2}+\dots\right\}.$$

(iii.) Two equal molecules:

$$U = -\frac{\pi\rho y_1^2 T^2}{\gamma}\left\{\frac{\sigma_1 c_1}{T^2-p_1^2}+\frac{\sigma_2 c_2}{T^2-p_2^2}+\frac{\sigma_3 c_3}{T^2-p_3^2}+\dots\right\}^2.$$

We are thus led to the following conclusions: — If the ether round two molecules be vibrating with the same period, and (a) this period be very large as compared with the atomic periods, then the intermolecular force (since $T = l/v$) will vary inversely as the wave-length squared; but, if this period be (b) very small as compared with the atomic periods, then the intermolecular force varies directly as the wave-length squared. Whether the force be attractive or repulsive depends, in general, on the sign of such expressions as

$$\frac{\sigma_1 c_1}{T^2-p_1^2}+\frac{\sigma_2 c_2}{T^2-p_2^2}+\frac{\sigma_3 c_3}{T^2-p_3^2}+\dots,$$

and this will vary according to the value of T relative to the p's. In addition, the sign of the b's involved in the c's is undetermined, and might possibly even vary from atom to atom. The *sense* of the force depends, therefore, on the physical constitution of the atom, and the relation between its free periods and that of the excited ether.

The above considerations would lead us to the conclusion that, when the molecules of a substance can move freely (*e.g.*, in a gas), a vibration of the ether (*e.g.*, a wave of light or heat) will bring into play intermolecular forces following laws very similar to those of statical

electricity.* A remark must here be made with regard to the case when the molecules are of the same structure. Since the bracket under case (iii.) is now squared, it might appear that two equal molecules under the action of an ethereal disturbance of the kind imagined could only repel each other. There are, however, one or two further points to be noted. If we had taken a disturbance of the type

$$C \sin 2\pi \left(\frac{t}{T} + a\right)$$

for y, so that $$y_1 = \frac{T}{2\pi}\frac{dy}{dt} = C \cos 2\pi \left(\frac{t}{T} + a\right)$$

for one molecule, and a *different* disturbance

$$C' \sin 2\pi \left(\frac{t}{T'} + a'\right)$$

for the other, so that $y_1' = C' \cos 2\pi \left(\dfrac{t}{T'} + a'\right)$;

then the mean of y_1, y_1' for a great number of vibrations would be zero, and there would be no term resulting in the force-function. The solitary exception to this rule is when $T' = -T$, that is, we are to treat the disturbance as in opposite phase. Thus, if we produce disturbances in the ether of the same periods but opposite phases round two molecules of like construction, it is possible that an attractive force may be called into play. The sense of the force called into play in two molecules by vibrations in the neighbouring ether will thus depend on two expressions of the form

$$y_1 T \left\{ \frac{\sigma_1 c_1}{T^2 - p_1^2} + \frac{\sigma_2 c_2}{T^2 - p_2^2} + \frac{\sigma_3 c_3}{T^2 - p_3^2} + \ldots \right\}$$

having the like or unlike signs.

By treating such expressions as representatives of positive and negative electricity, we are able to form a complete theory of statical electricity. The difference between positive and negative electricity becomes merely a relative one, and ether vibrations of the same period can produce positive electricity in one, and negative electricity in a second material (*cf.* Lindemann, p. 26). Lindemann deduces a

* Possibly herein lies the explanation of the motion of dust particles in a sunbeam, and of the radiometer.

similar result from his theory, but the principle on which he bases it does not seem to me very satisfactory:

"Two electrically excited particles attract each other if in one of them the electrical energy is, under the surrounding circumstances, capable of an increase. In the opposite case there is repulsion."

There is, of course, nothing to hinder our dealing with pulsating atoms in the same way, and attributing the difference between positive and negative electricity to a limit in the energy capacities of our atoms.

11. What we have said above of electrical disturbances in the ether producing the phenomena of statical electricity might be applied to ethereal vibrations produced by the sun, and these vibrations be used (instead of the free vibrations of p. 113 of the *Camb. Phil. Trans.* paper) to explain solar gravitation. In other words, the sun, by means of waves in the extreme ultra-violet (or waves whose lengths are extremely small as compared with the atomic diameter), might produce in the atoms a gravitation pulsation, which would suffice to explain attraction towards the sun. The question of inter-planetary gravitation would require some further consideration, as several difficulties arise with regard to its *sign*.

III. *Potential of two Electric Currents.*

12. An electric current must be looked upon as some form of transfer of molecular energy. We have dealt with statical electricity as pulsatory energy produced by the condition of the external medium. We have, then, now to consider the transfer of pulsatory energy along a chain of atoms or molecules. It is usual to suppose there is at every instant as much "positive" as "negative" electricity in each element of the current (*cf.* Lindemann, p. 29); but, until we have dealt more fully with the action of two atoms on each other, whether by impact or near approach, we cannot exactly determine how they transfer their pulsatory energy. We shall assume, however, that the elementary basis of a current is a pair of atoms with pulsations in *opposite phase* (*i.e.*, representing positive and negative electricity), moving in opposite senses in the direction of the current. Further, the relative motion of two atoms of different currents will be supposed small as compared with the velocity (v) of propagation of disturbance in the ether. We must examine the basis of our assumptions a little more closely. There seems no reason for treating the passage of electricity through a metal in a different manner from that in which

we treat the passage of electricity through a gas or electrolyte.* We are then led to look upon every electric current as accompanied by a molecular disassociation. In the simplest case which we can compare with an electric current, we have a row of diatomic molecules A_1B_1, A_2B_2, A_3B_3, A_4B_4, etc., which are individually disassociated and recombined as B_1A_2, B_2A_3, B_3A_4, etc.

This process, then, again repeats itself, so that A_2B_2, A_3B_3, A_4B_4, etc. again recombine; what happens to the final A and B, whether they are recombined or not, may well depend on the nature of the material through which the current passes, as well as the conditions at the terminals. We have seen (*Camb. Phil. Trans.* paper, § 32) how it is possible for an increase in the pulsatory energy of one atom (say A_1) to break up a diatomic molecule A_1B_1, and again [p. 39, case (ii.), (*b*)], for B_1, on joining A_2B_2, to turn out B_2. Thus, the changes we suppose the electric current to consist in, do not seem improbable on our theory, supposing we look upon electricity as itself pulsatory energy. Lindemann assumes the internal electrical energy of the molecule to consist in the energy of certain of its critical vibrations, whose periods, however, are extremely small as compared with those which we associate with light and heat (pp. 24—28). There is nothing to hinder our assuming that our atoms are capable of pulsations of like period (*Camb. Phil. Trans.* paper, p. 117). We are thus led to look upon two pulsating spheres moving in opposite senses in the direction of the current, as the basis of kinetic electricity. The mean speed of these two spheres must be equal, otherwise a steady process of disassociation and recombination would not be possible. This velocity will be that of the current of the negative electricity in one, and the positive electricity in the other direction.

In order to determine, therefore, the effect of one current upon a second, we must start by considering the influence of one pair of such vibrating pulsating atoms upon a second pair. The translations as well as the pulsations must, according to our theory, produce disturbances in the fluid ether, and so apparent systems of forces between the four atoms. But the terms in the force-function due to the former, besides being of a higher order in the ratio *diameter* : *distance*, are also much smaller than the latter, owing to our assumption that the ratio *relative velocity of atoms* : *velocity of ether disturbance* is a small quantity.

Let u_1 be the mean velocity of translation of an atom in one current, and u_2 the mean velocity in the other. Let ϕ_0 and ϕ_0' be the two atomic

* *Cf.* J. J. Thomson, "Applications of Dynamics to Physics and Chemistry," pp. 289—296.

pulsatory velocities. Then, as in Art. 10, we have the force-function for *free* pulsations:

$$U = +\frac{4\pi a_1^2 a_2^2}{\gamma}\rho\dot{\phi}_0\dot{\phi}_0'.$$

Now, ϕ_0 will be of the form

$$C\sin\left(\frac{2\pi t}{T}+a\right), \text{ where } T=\frac{\lambda}{v},$$

so that

$$\dot{\phi}_0 = \frac{2\pi v}{\lambda}C\cos\left(\frac{2\pi t}{T}+a\right).$$

In order that there may be a finite value for U, the pulsations must be of the same period, so that finally U takes the form (giving the mean value to the cosines)

$$U = +\frac{\beta\rho a_1^2 a_2^2 CC' v^2}{\lambda^2\gamma} \quad \text{.................... (xxv.)},$$

where β is a numerical constant, and we must change the sign of the term if the atoms have pulsations of opposite phase, *i.e.* opposite electricity.

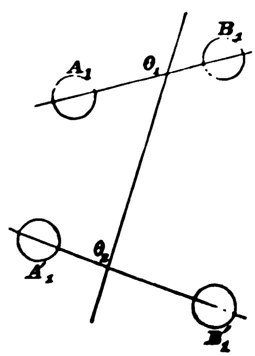
The question now arises, what changes must be made in λ owing to the fact that the atoms are in motion? Let V be the relative velocity of two atoms in the line joining their centres, $\lambda+\delta\lambda$ the changed wave-length; then, by the same principle as the change in the wave-length of the supposed hydrogen lines in Sirius is calculated, we have

$$\frac{\delta\lambda}{\lambda} = \frac{V}{v},$$

(see Herschel's *Outlines of Astronomy*, edition 1875, p. 702, and Lindemann, p. 29).

We easily find $\dfrac{1}{(\lambda+\delta\lambda)^2} = \dfrac{1}{\lambda^2}\left\{1-2\dfrac{V}{v}+3\dfrac{V^2}{v^2}\right\}$

if we neglect $(V/v)^3$.



Now, let θ_1, θ_2 be the angles between ds_1, ds_2, two elementary lengths in directions of translation of the atomic couples A_1, B_1 and A_1', B_1'. Then for A_1, B'_1, we have

$$V = u_1\cos\theta_1 - u_2\cos\theta_2,$$

and C and C' of same sign; for A_1, B_1' we have

$$V = u_1\cos\theta_1 + u_2\cos\theta_2,$$

and C, C' of opposite sign; for B_1, A_1' we have

$$V = -u_1\cos\theta_1 - u_2\cos\theta_2,$$

and C, C' of opposite sign; and, finally, for B_1, B_1' we have

$$V = -u_1 \cos\theta_1 + u_2 \cos\theta_2,$$

and C, C' of same sign. Thus the total force-function is given by

$$U = \frac{\beta\rho a_1^2 a_2^2 CC' v^2}{\lambda^2 \gamma} \left\{ 1 - \frac{2(u_1\cos\theta_1 - u_2\cos\theta_2)}{v} + \frac{3(u_1\cos\theta_1 - u_2\cos\theta_2)^2}{v^2} \right.$$

$$-1 + \frac{2(+u_1\cos\theta_1 + u_2\cos\theta_2)}{v} - \frac{3(u_1\cos\theta_1 + u_2\cos\theta_2)^2}{v^2}$$

$$-1 + \frac{2(-u_1\cos\theta_1 - u_2\cos\theta_2)}{v} - \frac{3(u_1\cos\theta_1 + u_2\cos\theta_2)^2}{v^2}$$

$$\left. +1 - \frac{2(-u_1\cos\theta_1 + u_2\cos\theta_2)}{v} + \frac{3(-u_1\cos\theta_1 + u_2\cos\theta_2)^2}{v^2} \right\};$$

or
$$U = \frac{\beta\rho a_1^2 a_2^2 CC'}{\lambda^2\gamma} \times (-24\cos\theta_1\cos\theta_2) \times u_1 u_2$$

$$= -\frac{\beta'\rho a_1^2 a_2^2 CC' u_1 u_2}{\lambda^2\gamma} \cos\theta_1 \cos\theta_2.$$

Now, every such atomic pair in an element of the current has to be taken with every such atomic pair in an element of the second current. If γ be now taken as the mean distance of the two elements, and we write*

$$\frac{1}{\sqrt{2}\lambda} \sqrt{\beta'\rho}\, \Sigma C a_1^2 u_1 = i_1 ds_1,$$

$$\frac{1}{\sqrt{2}\lambda} \sqrt{\beta'\rho}\, \Sigma C' a_2^2 u_2 = i_2 ds_2,$$

we have, for the *mutual potential* $(= -U)$ of two elements distant r,

$$\frac{2 i_1 i_2 ds_1 ds_2}{r} \cos\theta_1 \cos\theta_2 \ldots\ldots\ldots\ldots\ldots\ldots (\text{xxvi.}).$$

Hence for *closed* currents we may replace $\cos\theta_1\cos\theta_2$ by $\cos\epsilon$, where ϵ is the angle between the elements ds_1, ds_2, and we have for

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* Here u_1, u_2 are the mean velocities of separation in the pairs in either current element respectively. It might at first sight be supposed that these mean values are zero, as the atom B_1 oscillates from A_1 to A_2 with the process of disassociation and re-association; but we must really fix our eyes on the disassociating pairs, at first A_1B_1 and then B_1A_2, a great number of such couples being in each element. Thus u_1, u_2 will be the mean velocities of disassociating couples, and so always positive.

the mutual potential of two currents

$$2i_1 i_2 \iint \frac{\cos \epsilon}{r} ds_1 ds_2 \text{ .... (xxvii.).}$$

The result (xxvi.) is not absolutely accurate. It will be found that the terms of the third order in u/v vanish, so that we really are neglecting terms of the order $(V/v)^4$. The ratio of the propagation of electricity in wires to the propagation of light is, however, not a very small quantity, and it may be questioned whether we may legitimately neglect the fourth power of this ratio. We must content ourselves, then, with the remark that (xxvi.) probably gives the principal term in the mutual potential of two elements. Further, the result (xxvi.), while not agreeing with Ampère's form, is that deduced by Weber. Now, Helmholtz has shown that Weber's law admits of two particles starting with finite velocities, but that ultimately an infinite velocity can be reached. This case is, however, excluded by our treatment, for formula (xxvi.) no longer holds when we cannot neglect the fourth power of v/V. Hence Helmholtz's objection does not apply to the steps by which we have reached the generally accepted formula (xxvii.). This point has been fully dealt with by Lindemann in his treatment of the same problem; see his Memoir, pp. 31–33, and also Clerk-Maxwell's *Electricity and Magnetism*, §§ 852–855.

13. A further point is also brought out by Lindemann (pp. 34–35), which may be noted here, as it applies equally well to our system. The motion of a pulsating atom, if sufficiently great, changes the wave-length of the disturbance which its pulsation produces in the ether. It is, therefore, possible for waves of electric disturbance to appear as *light*, if the wave-length be sufficiently modified. Lindemann uses this result to explain electrical glow and the phenomena in Geissler's tubes. It will be seen that this explanation in no way depends upon the peculiar construction of Thomson's shell molecule, but is also true for the pulsating spherical atom.

IV. *On the Mutual Potential of Two Magnetic Molecules.*

14. Hitherto we have considered only pulsations, but it must be remembered that *polar* vibrations can also take place in our spherical molecules. The apparent mutual effect of polar vibrations in the direction of the common central distance of two molecules has been calculated for the first typical polar vibrations up to terms of the order $\left(\frac{\text{diameter}}{\text{central distance}}\right)^5$ in my first paper on this atomic hypothesis, *loc. cit.* p. 28. These results, which may be applied to the

kinetic theory of gases, are here of no value, because they refer only to polar vibrations in the *common central distance.* If we wish to attribute magnetic action to a polar vibration, we must, in order to obtain the most general formula, suppose the spherical atoms to have polar vibrations in perfectly general directions. Let these directions be h_1 and h_2. The most simple vibration we can imagine [after the pulsation represented in my first paper by $\dot{\phi}_0 P_0(\mu_1)$] is given by the surface displacement $\dot{\phi}_1 P_1(\mu_1)$, where μ_1 is taken about the direction h_1, and $P_1(\mu_1)$ is the first Legendre's coefficient (*Camb. Phil. Trans.* paper, p. 78). If $\mu_1 = \cos\theta_1$, this is of the form

$$C_1 \cos(n_1 t + a_1) \cos\theta_1.$$

Now, in the paper referred to, the term in the kinetic energy arising from such polar vibrations in two atoms is only calculated for the case of h_1 and h_2, both coinciding with the common central distance. We can, however, easily find it for the general case by noting that the radial distortion of a sphere given by $d_1 = c_1 \cos\theta_1$, is, if c_1 be small, equivalent to moving the sphere as a rigid body through the small distance c_1 in the direction of the polar axis h_1. Thus the polar vibrations $\dot{\phi}_1 P_1(\mu_1)$ and $\dot{\phi}'_1 P_1(\mu_2)$ are equivalent to translational vibrations $\dot{\phi}_1$ and $\dot{\phi}'_1$. Hence there is a term in the kinetic energy of two atoms with polar vibrations of the first kind of the form

$$-\pi\rho\dot{\phi}_1\dot{\phi}'_1 a_1^3 a_2^3 \frac{d^2}{dh_1\,dh_2}\left(\frac{1}{\gamma}\right)$$

(See Art. 2 of *Camb. Phil. Trans.* paper).

If, instead of endowing our atoms with internally maintained polar vibrations $\dot{\phi}_1$ and $\dot{\phi}'_1$, we suppose the motion of the ether to excite these vibrations, then we *must change the sign of this term*, and consequently two atoms upon which the ether forces polar vibrations will have an apparent *mutual potential*,

$$= -\pi\rho\, a_1^3 a_2^3 \dot{\phi}_1 \dot{\phi}'_1 \frac{d^2}{dh_1\,dh_2}\left(\frac{1}{\gamma}\right).$$

That is to say, they will act exactly as two magnetic elements if their vibrations are of the same period—the poles of the two atoms which are at the same instant elevated or depressed, being of the same magnetic sense, positive or negative.

Thus any disturbance of the ether which produces atomic vibrations of a polar kind is the basis of a magnetic field, and the magnetic effects will last as long as the field is maintained.

A permanent magnet must, in a certain sense, be a body which produces its own "field." This field may possibly be due to the pulsatory, or to the translatory vibrations of the whole system, or

even to the impact of the atoms. Thus we should expect changes of temperature to largely influence "permanent" magnetism.

The explanation of diamagnetism may possibly be attempted as follows: When the polar vibrations are forced on the atoms by the field, then the substances are paramagnetic; but, if the atoms have a free polar vibration which creates the field, then they are diamagnetic. It is possible that, in some cases, there are at the same time free and forced vibrations, and then the sign of the difference of the two terms in the apparent force-function must determine their paramagnetic or diamagnetic nature.

Bjerknes' explanation of diamagnetism is apparently different from this; see *Nature*, Vol. xxv., pp. 273–4; but I do not clearly follow it as described in the notice cited. Lindemann's explanation is given on pp. 45–7 of his Memoir; it involves assumptions which I think might be made as well for a pulsating spherical atom as for a Thomson's shell atom, but I do not feel entirely master of Lindemann's reasoning.

15. Lindemann devotes § 17, pp. 39–45, of his paper to the discussion of the rotation of the plane of polarised light produced by the electro-magnetic field. The only part of this theory which depends on Thomson's shell molecule are the second terms of the right-hand sides of equations (40) and (40*a*) of p. 42. These terms, however, correspond exactly to those we have introduced into equation (v.) of the present paper. Thus, if the argument by which Lindemann introduces the term $\frac{B}{2\pi}\frac{d^3z}{dx\,dt^2}$ into his equation for the ether vibration be really a valid one, it applies just as well to the pulsating atom as to the shell atom, and Lindemann's deduction of the rotation of the plane of polarisation holds, with all his conclusions, as well for the pulsating atom as for Thomson's.

But, after careful examination of Lindemann's reasoning on p. 41, I must confess that I do not feel by any means satisfied by the process whereby he deduces the presence of the term $\frac{B}{2\pi}\frac{d^3z}{dx\,dt^2}$ in the ether equation. Nor, without further investigation of the exact action of pulsating spheres on the ether fluid, am I prepared to offer a really satisfactory explanation of the rotation of the plane of polarisation, any more than I am prepared with an explanation of double refraction. In both cases I can only say that Lindemann's reasoning appears to be quite independent of the particular hypothesis of Thomson's shell atoms, but at the same time the reasoning itself does not seem to me sufficiently convincing to be adopted as a real explanation of the phenomena.

16. The action of a pulsating sphere on another sphere moving with velocity q in direction h, and at a distance γ, is approximately given by the term in the force-function

$$2\pi\dot{\phi}_0 q a_1^2 a_2^3 \rho \frac{d}{dh}\left(\frac{1}{\gamma}\right).$$

Replacing the single pulsating sphere by two disassociating with mean velocity u, we have, as in Art. 12, a term of the form

$$-\frac{\beta'' \rho a_1^2 a_2^3}{\lambda} u \cos\theta\, q \frac{d}{dh}\left(\frac{1}{\gamma}\right),$$

where $\cos\theta$ is the angle between γ and the direction of the current pair. If we sum for all pairs of the element and all elements of the current, we should expect a term in the kinetic energy, or in the apparent force-function of the current on the vibrating sphere, of the form

$$-Bi\,q.$$

Thus, in the direction x there would be a force on the atom

$$= -Bi\frac{dq}{dx}.$$

As a result, there would be a force exercised by the atom on the ether of the same form. Now, if we suppose the atom to be vibrating in the direction z, perpendicular to x, $q = dz/dt$, and the force becomes

$$-Bi\frac{d^2z}{dx\,dt}.$$

In finding the effect of a number of such molecules, supposed to have the same velocity (or a proportional velocity), to the ether in their immediate neighbourhood, we must obviously take the difference with regard to x of such force as the above, and we might conclude that a term of the form

$$Ai\frac{d^3z}{dx^2\,dt}$$

would be introduced into the equation for the ether. Now, suppose the ether to be vibrating as well in direction y as in direction z, then we must modify equation (v.) of Art. 2, by the introduction of two terms such as the above. We find, since $\phi_s = B_s y$, if $\Sigma c_s B_s = \phi(T)$,

$$\rho\frac{d^2y}{dt^2} = k\frac{d^2y}{dx^2} + \{f+\phi(T)\}\,y + Ai\frac{d^3z}{dx^2\,dt} - A'i\frac{d^3y}{dx^2\,dt},$$

$$\rho\frac{d^2z}{dt^2} = k\frac{d^2z}{dx^2} + \{f+\phi(T)\}\,z - Ai\frac{d^3y}{dx^2\,dt} + A'i\frac{d^3z}{dx^2\,dt}.$$

This supposes the medium to be isotropic round x. The term with the coefficient A is similar to that introduced by Maxwell (p. 827 of his *Electricity and Magnetism*), and, if A' were zero, these equations would, I think, suffice to explain the rotation of the plane of polarisation. In order, however, that a rotation round the axis of x of the axes y, z, may not alter the form of these equations, it is necessary that $A' = o$.

It should be noted that, whereas the above investigation leads to a term of the form $\frac{d^3 z}{dx^2\, dt}$, Lindemann's reasoning introduces one of the form $\frac{d^3 z}{dx\, dt^2}$.

17. The results of this paper, although exhibiting much need for further inquiry, will, I think, suffice to show that the results obtained by Lindemann are in no way dependent on the structure of the shell atom of Thomson; that, further, the pulsating spherical atom is better adapted than the shell atom to explain chemical association and disassociation, while, as shown in my first paper, it succeeds in throwing a certain amount of light on chemical, cohesive, and even gravitating force.

I propose, in a third paper, to deal with the subjects of elasticity and cohesion. The peculiar features of the generalised elastic equations arise from the fact that intermolecular action, and therefore the internal work of an elastic strain, depends upon the pulsatory and translational energy of the molecules—thus a sufficiently rapid vibration may introduce terms depending on the relative velocity of adjacent parts of the material into the expression for the work due to the instantaneous strain. The modified equations will, I think, tend to throw light on several problems connected with optics and with elastic after-strain.

On Cyclotomic Functions. By H. W. Lloyd Tanner, M.A., Professor of Mathematics in the University College of South Wales.

[*Read Nov. 8th*, 1888.]

The paper of which the first two sections are now submitted to the Society, communicates the results obtained in a research which was suggested to me by a remark of Prof. Sylvester, in his "Excursus on Cyclotomic Functions." Section I. is devoted to an examination of the theory of groups of the numbers prime to n and less than n (= the totitives of n). This theory seems to be of considerable interest for its own sake; but, in the present paper, it is studied with

a view to its application to cyclotomic functions. The main problems are—Given a group of totitives, to determine the simple groups by the multiplication of which it is produced; and, secondly, to determine all the groups which are comprised in a given group. The theory is based upon the known theory of groups of substitutions or operations, from which, however, it differs in some interesting particulars, and from this theory most of the nomenclature and method is derived.

In Section II., I venture to propose a more general definition of a period of roots of unity. The group-theory is useful in discussing these periods, which are shown to have the fundamental properties of the periods of Gauss and Kummer.

In the remaining sections of the paper, which I hope to present at an early meeting, I propose to deal with certain properties of cyclotomic functions, which seem to have escaped notice.

Section I.

Groups of Totitives of n.

1. A *group of numbers* is a collection of numbers which includes every positive integral power and every product of its constituents. For example, the powers of a number a form a group of which a may be called the *base*. Groups may have an infinite number of bases. For instance, the numbers $\lambda a+1$, where λ takes all the values $0, 1, 2, \ldots$, make up a group; and, since all the prime numbers included in the group must be bases, there are an infinite number of bases.

2. If a collection of numbers is such that the product of every pair of elements in it (including the case in which the pair consists of the same element repeated) is included, the collection is a group. This gives a convenient test. That the numbers $\lambda a+1$ form a group, for example, is verified by observing that the product of any two numbers of this form is itself included in the form.

3. In a group of numbers, which is necessarily infinite, let all the numbers which are congruent with respect to a modulus n be regarded as equivalent. Or, what is in effect the same thing, let every number in a group be replaced by its minimum positive residue to modulus n. The resulting aggregate is a finite group of residues of n.

As examples of such groups may be mentioned the group of numbers less than n and prime to n, or, to use Sylvester's convenient name, the totitives of n; the numbers which have a certain factor a in common with n, but contain no factor of n which is prime to a (the a-totitives of n, Mitchell); the r^{th} power residues of n. That these are groups follows at once by applying the test of Art. 2.

Mixed Groups.

4. The theory of groups of residues of n differs in certain points from the theory of groups of substitutions or operations. One characteristic difference will be noted, since it is of use in the sequel.

Let a, b be two prime factors of n, and consider a group G comprising some multiples of a, some of b, and therefore some of ab. These elements are assumed to complete the group, and none of them is divisible by any prime factor of n, except a or b. For example, if $n = 60$, $a = 3$, $b = 5$, the numbers 3, 5, 9, 15, 21, 25, 27, 45 make up such a group.

Now, in G the a-totitives form a group of themselves, for every product of two of the a-totitives is present in G, and is itself an a-totitive. So the b-totitives form a group, and the ab-totitives form a third group. Hence the group G is made up of the three groups by addition. The group given above may be written

$$(3, 9, 21, 27) + (5, 25) + (15, 45),$$

where each ( ) includes a group.

Groups formed by addition in this way are in some respects analogous to the "mixtures" or "mixed substances" of the chemists, and will be called *mixed groups*.

The r^{th} power residues of n give other examples of these mixed groups. The quadratic residues of 15, for instance, are

$$(1, 4) + (6, 9) + (10) + (0),$$

and the biquadratic residues are

$$(1) + (6) + (10) + (0),$$

where, as before, each ( ) includes a group.

5. Any group of the totitives of n is an unmixed or pure group. For every group of totitives must contain the element 1. Hence, if we subtract from a group of totitives any group contained in it, the remaining numbers do not include 1, and therefore do not make up a group.

Order of a Group, Order of an Element. (Arts. 6—10.)

6. The order of a group of totitives is the number of elements it contains. It is convenient also to speak of the order of an element of the group. We shall say that the order of an element a is α, when $a^{\alpha} \equiv 1$, mod. n, and no lower power of a satisfies this congruence.

The meaning is the same as that of the statements—"a belongs to the exponent α," "a is a primitive α^{th} root of unity to modulus n." The innovation is, I hope, justifiable, partly because it helps to mark the analogy of the present theory to the theory of groups of substitutions, and partly because of its convenience in use.

7. The order of any power of an element a is a factor of the order of a. For, if α, ξ be the orders of a, a^{λ}, ξ is the smallest number which makes $\lambda\xi$ a multiple of α; that is, $\xi = \alpha/\alpha'$, where α' is the greatest common measure of λ and α. Amongst the powers of a there is (at least) one element whose order (β) is an assigned factor of the order (α) of a. For, if $\alpha = \beta.\gamma$, a^{γ} is such a power.

8. Consider, next, two elements, a, b, whose orders are α, β respectively. The order of the product ab is some factor of the least common multiple (say γ) of α, β. For $(ab)^{\gamma} \equiv 1$, mod. n. If α, β are relative primes, the order of ab is $\alpha\beta$. For, if ξ be the order of ab,

$$a^{\xi}.b^{\xi} \equiv 1, \text{ mod. } n,$$

therefore $$a^{\xi\beta}.b^{\xi\beta} \equiv 1, \quad ,,$$

therefore $$a^{\xi\beta} \equiv 1, \quad ,,$$

so that $\xi\beta$ is a multiple of α, and, since β is prime, $\alpha\xi$ is a multiple of α. Similarly, it is a multiple of β, and therefore of $\alpha\beta$. As it is also a factor of $\alpha\beta$, it follows that $\xi = \alpha\beta$.

9. If in a group all the elements of orders α and β are given, where α, β are prime to each other, then all the elements of order $\alpha\beta$ are implicitly given. For, if c be any element of the group whose order is $\alpha\beta$, c^{β} and c^{α}, being elements of orders α, β respectively, are given; say, they are a, b. Then, for all values of λ, μ,

$$c^{\lambda\beta+\mu\alpha} = a^{\lambda}.b^{\mu}.$$

But, since α and β are relative primes, λ, μ may be determined so that

$$\lambda\beta+\mu\alpha \equiv 1, \text{ mod. } \alpha\beta,$$

and c is expressed in terms of the data.

10. An important corollary to this is that a group is completely specified by those of its elements whose orders are primes or powers of primes.

Independent Elements and Groups. (Arts. 11—14.)

11. Two elements which have no power except 1 in common are called independent elements. The definition may also be expressed

by saying that a, b are independent elements if the congruence

$$a^{\lambda}b^{\mu} \equiv 1, \text{ mod. } n,$$

necessarily implies the two congruences

$$a^{\lambda} \equiv 1, \quad b^{\mu} \equiv 1, \text{ mod. } n.$$

It is an immediate consequence of this that, when a, b are two independent elements of orders α, β respectively, the order of ab is the least common multiple of α, β.

For, if ξ be the order of ab,

$$a^{\xi} . b^{\xi} \equiv 1, \text{ mod. } n,$$

therefore $$a^{\xi} \equiv 1, \quad b^{\xi} \equiv 1, \text{ mod. } n,$$

therefore ξ is multiple of α and of β.

In particular, if β is a factor of α, the order of ab is α.

The converse is also true when α, β are prime to each other; viz., any two elements whose orders are relative primes are independent elements. For, if

$$a^{\lambda} . b^{\mu} \equiv 1, \text{ mod. } n,$$

therefore $$a^{\lambda\beta} . b^{\mu\beta} \equiv 1, \quad ,,$$

therefore $$a^{\lambda\beta} \equiv 1, \quad ,,$$

Hence $\lambda\beta$, and therefore λ, is a multiple of α; so that

$$a^{\lambda} \equiv 1, \text{ mod. } n.$$

From which it follows that

$$b^{\mu} \equiv 1, \text{ mod. } n.$$

12. The elements $a, b, \ldots c$ are termed independent if the congruence

$$a^{\lambda} . b^{\mu} \ldots c^{\nu} \equiv 1, \text{ mod. } n,$$

cannot be satisfied unless all the congruences

$$a^{\lambda} \equiv 1, \quad b^{\mu} \equiv 1, \ \ldots, \ c^{\nu} \equiv 1, \text{ mod. } n,$$

are satisfied.

The same may be stated in a different form; viz., that the congruence

$$a^{\lambda} . b^{\mu} \ldots c^{\nu} \equiv a^{\lambda'} . b^{\mu'} \ldots c^{\nu'}, \text{ mod. } n,$$

implies all the congruences

$$a^{\lambda} \equiv a^{\lambda'}, \quad b^{\mu} \equiv b^{\mu'}, \ \ldots \ c^{\nu} \equiv c^{\nu'}, \text{ mod. } n.$$

13. A set of elements $a, b, \dots c$ is an independent set if the orders of every pair are prime to each other. For, if

$$a^{\lambda} . b^{\mu} \dots c^{\nu} \equiv 1, \text{ mod. } n,$$

then
$$a^{\lambda\pi} . b^{\mu\pi} \dots c^{\nu\pi} \equiv 1, \text{ mod. } n.$$

Now, take π to be the product of the orders, α, β, &c., of all the elements a, b, &c., except one, say except c. Then each of the factors $a^{\lambda\pi}$, $b^{\mu\pi}$, ... is unity, so that

$$c^{\nu\pi} \equiv 1, \text{ mod. } n,$$

and
$$\nu\pi \equiv 0, \text{ mod. } \gamma,$$

where γ is the order of c. But, by supposition, π is prime to γ, so that

$$\nu \equiv 0, \text{ mod. } \gamma,$$

and
$$c^{\gamma} \equiv 1, \text{ mod. } n.$$

As c is any one of the elements, the theorem is proved.

14. Groups are called independent if every set of elements that can be formed by taking one element from each group is an independent set of elements.

For two groups, it is a sufficient test of independence to examine whether they have any elements, other than 1, in common.

Products and Quotients of Groups. (Arts. 15—20.)

15. The product of two or more groups is the aggregate of all the products of the elements of the several groups, one from each group. It is easily seen that this aggregate is a group. If the factor groups are independent, the order of the product is the product of the orders of the factors, since (Art. 12) all the elements in the product are different.

16. If a group F contains a group G, the quotient F/G may be formed in the following manner.* Make a table in the first row of which are entered all the elements of G, while in the following rows occur the products of these elements into a, b, ..., elements of F,

$$\begin{array}{llll} G = 1, & a, & b, \dots & c, \\ \mathrm{a}G = \mathrm{a}, & \mathrm{a}a, & \mathrm{a}b, \dots & \mathrm{a}c, \\ \mathrm{b}G = \mathrm{b}, & \mathrm{b}a, & \mathrm{b}b, \dots & \mathrm{b}c. \\ \dots & \dots & \dots & \dots \end{array}$$

* The tabular arrangement adopted is taken from Netto, "Substitutionentheorie," p. 43.

All the elements in this array are elements of F. The elements in any row are all different, for $\mathrm{a}a \equiv \mathrm{a}b$, mod. n, implies $a \equiv b$, mod. n, since a is a totitive of n.

If a is an element of G, aG is identical with G with its elements differently arranged.

If a is not an element of G, then G and aG have no element in common. For $\mathrm{a}a \equiv b$ implies $\mathrm{a} \equiv a^{-1}b$, viz., that a is an element of G.

In the same way, the rows aG, bG are identical, or have no element in common, according as b is, or is not, included in the row aG.

To form the quotient F/G, we have to select a, any element of F not included in G; then b, any element not included in G or aG, and so on, until all the elements of F are exhausted. The quotient is (1, a, b, ...).

17. The effect of multiplying every element in the table by any element c, of F, is merely to rearrange the groups among themselves, and the elements of each group. But no element is transferred from one group to another; so that, essentially, the table is unchanged. For, aG becomes ca$G$, a row of the existing table which is or is not identical with aG according as ca is or is not an element of aG. The two rows ca$G$, cb$G$ are or are not identical according as aG, bG are or are not identical. And it is clear that no elements (and therefore no rows) are gained or lost by the change, since the group F cannot be changed by being multiplied throughout by one of its own elements.

18. It follows from Art. 16 that the order of F is a multiple of the order of G, the multiplier being the number of elements in the quotient (1, a, b, ...), and that the order of any group contained in F is a factor of the order of F.

19. Since the group of all the totitives of n has for its order the number of totitives of n [= the *totient* of n, $= \tau n$ (Sylvester)], the order of any group of totitives is a factor of τn.

20. Consider any element a of order α contained in a group F. Then the group of powers of a, viz.,

$$a,\ a^2,\ \dots\ a^{\alpha-1},\ a^{\alpha}\ (=1),$$

is included in F. This group is of order α. Hence the order of a group is a multiple of the order of each of its elements.

Decomposition of G. First Step. (Arts. 21—24.)

21. We now pass to the consideration of the decomposition of a given group G. The decomposition may either be a *proper* decomposition in which every factor is an independent group; or it may be *improper* by reason of the presence of some factors which are not groups; or, what is the same thing, the factors may all be groups, but these groups are not a set of independent groups.

22. From the group G select all the elements whose orders are the prime number p, and powers of this prime. These elements form a group, which we shall call G_p. For the product of any two of these elements are included in G, and it is also included in the selection, since its order is some power of p. (Art. 8.)

23. Let $p, q, \dots r$ be all the different primes that occur as factors of the orders of the elements of G. Then G contains elements whose orders are $p, q, \dots r$, or powers of these primes (Art. 7). And we have

$$G = G_p \cdot G_q \dots G_r,$$

where G_p has the same meaning as in the last article, and $G_q, \dots G_r$ have analogous significations. For the product on the right is a group which contains all those elements of G whose orders are primes or powers of primes, and therefore (Art. 10) must be identical with G.

24. Since the groups $G_p, G_q, \dots G_r$ form an independent set (Arts. 13, 14), the order of G is the product of the orders of $G_p, G_q, \dots G_r$ (Art. 15). Hence the determination of the order of G is reduced to the simpler case in which all the elements have for their orders powers of one and the same prime.

Simple Groups. (Arts. 25, 26.)

25. It may happen that G_p consists of powers of one of its elements a, and has no other elements. If the order of a be p^s, the order of the group is also p^s (Art. 20).

In this case G_p is called a *simple group*, so that a simple group is defined by the two characters:

(i.) The order of each of its elements is a power of one and the same prime.

(ii.) Each of its elements is a power of one of the elements.

26. The name *simple group* is given because such a group does not admit of a proper decomposition. If the order of the group be p^a and $a > 1$, the simple group S contains other groups; but the quotient neither is a group nor contains a group. Every representation of S as a product comprises only one factor which is a group, the other factor or factors being parts of groups.

The proof of these statements rests upon the fact that any group S of order p^λ, contained in S, contains in itself all the groups of lower order contained in S. In effect, if we select any element b of order p^λ, the group of which b is the base is of order p^λ, and therefore includes all the elements of S whose orders are not greater than p^λ. Hence the quotient S/S_λ contains, besides 1, only elements whose orders are greater than p^λ; it does not contain the group S, of order p, which is a necessary part of every group contained in S. Thus S/S_λ neither is nor contains a group.

Product of Groups which are not independent. (Arts. 27—29.)

27. Before considering the general case in which G_p is not a simple group, it is desirable to discuss the product of two or more simple groups of orders p^a, p^β, &c., which are not independent.

Taking first the case of two groups, say

$$A\,(1, a, a^2, \dots a^{p^a-1}), \quad B\,(1, b, b^2, \dots b^{p^\beta-1}),$$

and let a be not less than β.

The index of the lowest power of b which occurs in A must be a power of p. For any element of B can be written in the form b^{ip^γ}, where i is prime to p, and $\gamma < \beta$ (and therefore $< a$). If this power of b is an element of A, we have

$$a^j \equiv b^{ip^\gamma}, \text{ mod. } n,$$

so that

$$a^{lj} \equiv b^{lip^\gamma}, \text{ mod. } n.$$

Now, l can be taken so that

$$li \equiv 1, \text{ mod. } p^{\beta-\gamma},$$

and with this value of l the exponent of b becomes

$$p^\gamma + k\,.\,p^\beta,$$

so that

$$a^{lj} \equiv b^{p^\gamma}, \text{ mod. } n.$$

If, then, to γ be assigned its lowest value, p^γ is the index of the lowest power of b which is common to the two groups.

When $\gamma = 0$, b, and therefore every element of B, is included in A, so that the product is merely A.

When $\gamma > 0$, the products

$$A,\ bA,\ b^2A,\ \ldots$$

begin to repeat with the product

$$b^{p^\gamma}A = A;$$

for $$b^{k+p^\gamma}\,.\,A = b^k\,.\,b^{p^\gamma}\,.\,A = b^k\,.\,A.$$

Hence the quotient,

$$A\,.\,B/A = (1,\ b,\ b^2,\ \ldots\ b^{p^\gamma}-1).$$

Therefore, the product

$$A\,.\,B = A\ (1,\ b,\ b^2,\ \ldots\ b^{p^\gamma-1}),$$

and the order of the product is $p^{\alpha+\gamma}$.

28. This product can always be replaced by a product of simple independent groups.

From the congruence $a^{lj} \equiv b^{p^\gamma}$, mod. n,

we have, by raising both sides to the power $p^{\beta-\gamma}$, which is integral since $\beta > \gamma$,

$$a^{ljp^{\beta-\gamma}} \equiv b^{p^\beta} \equiv 1, \text{ mod. } n,$$

therefore $$ljp^{\beta-\gamma} \equiv 0, \text{ mod. } p^\alpha,$$

therefore $$lj = h\,.\,p^{\alpha-\beta+\gamma},$$

wherein the index of p is necessarily positive. Here h is prime to p; for, putting $h = h'p^\delta$, substitute in the first congruence of this article, and then raise both sides to the power $p^{\beta-\gamma-\delta}$: we get

$$b^{p^{\beta-\delta}} \equiv 1,$$

which requires $\delta = 0$.

Now, take a new base c defined by the congruence

$$bc \equiv a^{hp^{\alpha-\beta}}, \text{ mod. } n.$$

A solution of this can always be found, for the known elements b, $a^{hp^{\alpha-\beta}}$ are prime to n. Hence

$$b^\lambda\,.\,c^\lambda \equiv a^{\lambda hp^{\alpha-\beta}}, \text{ mod. } n.$$

If we assign to λ any value less than p^γ, b^λ (and therefore c^λ) is not included among the powers of a. If we put $\lambda = p^\gamma$, we have

$$b^{p^\gamma} . c^{p^\gamma} = a^{hp^{\alpha-\beta+\gamma}} = b^{p^\gamma}, \text{ mod. } n,$$

and therefore $$c^{p^\gamma} \equiv 1, \text{ mod. } n.$$

Thus c is a totitive of order p^γ, having none of its powers coincident with powers of a. The product

$$(1, a, a^2 \dots a^{p^\alpha-1})(1, c, c^2 \dots c^{p^\gamma-1}) = A . C, \text{ say},$$

is a product of two simple independent groups.

That this product is the same as the product $A . B$ is proved if we show that it contains the element b, for then it contains the group B.

But, since $$b = c^{-1} . a^{hp^{\alpha-\beta}},$$

it must be present in any group which contains a and c.

29. A similar process applies to the product of three or more groups which are not independent. For the sake of clearness we give the discussion for the case of three groups

$$A(1, a, a^2 \dots), \ B(1, b, b^2 \dots) . C(1, c, c^2 \dots),$$

which are of orders p^α, p^β, p^γ, respectively. It is unnecessary to consider the case in which pairs of the groups are dependent, for this merely requires a repeated application of the preceding articles. We assume, then, that the relation expressing dependence is of the form

$$a^\lambda . b^\mu . c^\nu \equiv 1, \text{ mod. } n,$$

where none of the factors is unity. Let C be the group whose order is not higher than that of the others. As in Art. 27, the lowest power of c which occurs in the product AB has for its index a power of p, say p^δ.

Thus $$c^{p^\delta} \equiv a^\lambda . b^\mu;$$

whence $$\lambda = h . p^{\alpha-\gamma-\delta},$$

and $$\mu = k . p^{\beta-\gamma-\delta},$$

and at least one of the coefficients h, k is prime to p. Now select d to satisfy the congruence

$$c . d \equiv a^{hp^{\alpha-\gamma}} . b^{kp^{\beta-\gamma}}.$$

Then d is a totitive of order p^t having none of its powers included in the group AB. And

$$A.B.C = A.B.D,$$

a product of three independent groups.

Decomposition of G_p. (Arts. 30, 31.)

30. The results just obtained enable us to prove, by actually effecting the decomposition, that G_p is always expressible as a product of simple independent groups.

Select any one of the elements, a, of G_p, and form the simple group A, which has a for its base. The group A is necessarily contained in G_p, and the quotient G_p/A is formed as in Art. 16. From this quotient an element b is selected whose group B, we will suppose, is comprised in the quotient so that A, B are independent groups. Proceeding in this manner, we separate a set of independent groups A, B, ... C contained in G_p. In dealing with the quotient at this stage, viz. $G_p/A.B \dots C$, suppose that the selected element d is not independent of a, b ... c; so that in the quotient the whole group D does not occur. This must be because some power of d is contained either in one of the groups already separated, or in the product of two or more of them. In either case the product with D may be modified by the process of Arts. 28, 29 so as to consist of a product of simple independent groups. We can proceed in like manner as long as the quotient contains any element besides 1. And it thus appears that the decomposition of G_p into a product of simple independent groups can always be effected.

31. Since the order of every simple group in G_p is a power of p, and the order of G_p is the product of the orders of the simple independent groups into which it is decomposable, it follows that the order of G_p must also be a power of p. Hence further, if the order of any group of totitives is m, then there are groups G_p, G_q, ... corresponding to every prime factor of m. For each group G_p contributes a power of p, and no other prime, to the factors of m.

32. In the decomposition of a group of totitives, G, there is evidently much that is arbitrary; but in some regards the decomposition is quite determinate. In the first place, the resolution into groups, G_p G_q, ..., corresponding to the prime factors p, q, ... of the order of G, is unique. In the second place, the number of simple groups of any proposed order is determinate. We prove this by *showing* how to calculate these numbers when the undecomposed

group is given. There is no loss of generality in confining the discussion to the case of one of the groups G_p.

Number of Simple Groups contained in G_p. (Arts 33–35.)

33. We determine, first, the number of elements of order not exceeding p^μ in the product of the simple independent groups $A, B \dots C$. Let the group of highest order A be of order p^λ, and suppose that there are h_i groups of order p^i ($i = 1, 2, 3 \dots \lambda$). Also suppose that there are k_i groups whose order is p^i at least; so that

$$k_i = h_i + h_{i+1} + \dots + h_\lambda,$$

and
$$h_i = k_i - k_{i+1}.$$

From any group such as A whose order is greater than p^μ take a reduced group A', comprising all the elements of A whose orders are not greater than p^μ. Form a product

$$A' . B' \dots C = R,$$

of all these reduced groups and of those groups, such as C, whose orders are not greater than p^μ. This reduced product contains all the elements whose number we seek, and no others.

In fact, the order of a product of independent elements $a, b \dots c$, of orders $p^\alpha, p^\beta \dots p^\gamma$, is p^α, if α is not less than $\beta \dots$ or γ. For this is true of binary products (Art. 11), and therefore of a product of any number of independent elements.

It follows that the product R contains only elements of order not exceeding p^μ. On the other hand, if any one of the elements excluded from R were present in any product, the order of that product would be greater than p^μ.

Now, R contains

$$\begin{array}{llll} h_1 & \text{groups} & \text{of} & \text{order } p, \\ h_2 & \text{,,} & \text{,,} & p^2, \\ \dots & \dots & \dots & \dots \\ h_{\mu-1} & \text{groups} & \text{of} & \text{order } p^{\mu-1}, \\ k_\mu & \text{,,} & \text{,,} & p^\mu; \end{array}$$

therefore the order of R, *i.e.* the number of elements in R, being the product of the orders of all its factors, is

$$= p^{h_1} . p^{2h_2} \dots p^{(\mu-1) h_{\mu-1}} . p^{\mu k_\mu}.$$

In other words, the number of elements of $A.B \dots C$ of order not exceeding p^μ is a power of p whose index is

$$= h_1 + 2h_2 + \dots + (\mu - 1)\, h_{\mu-1} + \mu k_\mu$$
$$= k_1 + k_2 + \dots + k_{\mu-1} + k_\mu = K_\mu, \text{ say.}$$

Note that $k_1 = K_1$ is the number of all the simple factors.

34. This gives a rule for finding the values of h for a given undecomposed group. Count the number of elements whose orders are not greater than $p, p^2, p^3 \dots$ respectively. These numbers are powers of p whose indices are $K_1, K_2, K_3 \dots$. The first differences, $K_\mu - K_{\mu-1}$, give the values of k_μ, and the second differences with changed sign,

$$k_\mu - k_{\mu+1}, = -K_{\mu+1} + 2K_\mu - K_{\mu-1},$$

give the values of h_μ.

For example, consider the group of all the totitives of 80. The totient of 80 is 32, so that each of the groups has some power of 2 for its order. By counting, we find that the numbers of elements whose

orders are not greater than	1,	2,	4,	8, respectively,
are	1,	8,	32,	32,
therefore	$K =$ 0,	3,	5,	5,
	$k =$	3,	2,	0,
	$h =$		1,	2,

Hence the group of totitives of 80 is a product of 3 simple independent groups ($k_1 = K_1 = 3$) of which 1 is of order 2, and 2 are of order 4.

Groups of Totitives for different Moduli. (Arts. 35, 36.)

35. There is another method of forming this framework of a group G which depends on the relations between simple groups of totitives of different moduli.

Let $$A\,(1, a, a^2 \dots)$$

be a group of order p^a of the totitives of lm, where l, m are relative primes. If we replace each of these elements by its residue to modulus l, we obtain a group

$$B\,(1, b, b^2 \dots)$$

of the totitives of l. For, if $a \equiv b$, mod. l,

then $$a^r = b^r, \text{ mod. } l.$$

Here b must be an element of order p^β, where $\beta = 0, 1, 2, \ldots$ or a.

Similarly, we may derive a group
$$C(1, c, c^2, \ldots)$$
of the totitives of m.

We shall prove that at least one of the groups B, C is of order p^a, the order of A.

When b^r, c^r are given, the element a^r is uniquely determined by the congruences
$$a^r \equiv lx + b^r \equiv my + c^r, \text{ mod. } lm,$$
since l, m are prime to each other.

Hence, if $$b^r \equiv b^s, \text{ mod. } l,$$
and $$c^r \equiv c^s, \text{ mod. } m,$$
then $$a^r \equiv a^s, \text{ mod. } lm,$$
and $$r \equiv s, \text{ mod. } p^a.$$

In particular, when $\lambda < a$, only one of the congruences
$$b^{p^\lambda} \equiv 1, \text{ mod. } l,$$
$$c^{p^\lambda} \equiv 1, \text{ mod. } m,$$
can be satisfied; for, taken together, they imply that a is of order p^λ. That is to say, if c is an element of order p^λ, b does not satisfy the congruence just written, and therefore its order is higher than p^λ. And in the same way, if b is an element of order p^λ, less than the order of a, c is of higher order than b. It follows that one of the elements b, c must be of order p^a; in other words, one of the groups B, C must be of the same order as A.

36. A corollary to this theorem is that, if
$$n = a^a \cdot b^\beta \ldots c^\gamma,$$
where $a, b, \ldots c$ are different primes, every simple group of totitives of n corresponds to a group of totitives of one of the factors $a^a, b^\beta \ldots c^\gamma$, and is of the same order as this group. Since the order of the n-group is the product of the orders of the groups belonging to a^a, &c., no groups are repeated or lost. Thus, if the totitives of a^a consist of a product of simple independent groups
$$A_1 \cdot A_2 \ldots A_\lambda,$$

the totitives of $b^\beta = B_1 \,.\, B_2 \dots B_k,$

and the totitives of $c^\gamma = C_1 \,.\, C_2 \dots C_l,$

then the totitives of n

$$= A_1' A_2' \dots A_h' B_1' \dots B_k' \dots\dots C_1' \dots C_l',$$

where A_1', for instance, is of the same order as A_1, but contains a different base determined by two congruences such as the first pair given in Art. 35.

A numerical example is given in Art. 44, *post.*

Enumeration of Groups which are Factors of G. (Art. 37.)

37. When the "framework" of a group decomposition has been determined, there is a very great variety in the ways of filling up. Any element of proper order may be taken as base of each simple group, subject to the single condition that all the bases must be independent. When improper decompositions are admitted, the variety is enormously increased.

Instead of considering the question from this point of view, it is convenient for our present purpose to indicate a method of obtaining, without omission, a table of all the groups which are factors of a given group. Since every group can be expressed as a product of simple independent groups, we have only to form a table of these; and, to avoid repetitions, the table should indicate when a set of groups is independent. It is clear that we may consider the groups G_p separately.

One way of making such a table will now be indicated. It depends upon the fact that two groups included in G_p are, or are not, independent according as they have or have not in common a group of order p. Select any element of order p, and enter its group in a line of the table. Similarly deal with one of the remaining elements of order p, and so on until all these are exhausted. Each group is entered on a separate line. Taking any element of order p^2, form its group, and enter it on the same line as the p-group which it contains. The table is filled in this way with groups, every group on the same line as the group of order p which it contains.

The table as it stands gives all the simple groups of G_p.

To form the binary products, we may take any group in one line, and multiply it by any group in another line.

For products of three groups the three lines must be independent, and this is secured if the groups of order p are independent. The like precaution is needed for products of more than three groups.

In this way, we get all the groups contained in G_p; but not without repetition. If, for instance, we try to form the product of highest possible order, there are, in general, several ways of doing it; but there is only one such product, viz. G_p itself. In the comparatively simple cases I have tried, the trouble thus caused has not been very great.

An example of such a table is given in Art. 46.

Monobasic Groups. (Arts. 38—41.)

38. A group consisting of a single element and its powers is called a *monobasic group*, and the element named is its base. By definition all simple groups are monobasic, but the converse is not true. For, if a, b are any two elements whose orders α, β are prime to each other, then ab is an element of order $\alpha\beta$ (Art. 8), and the product of the groups

$$(1, a, a^2, \ldots)\,(1, b, b^2, \ldots)$$

is a monobasic group, the base being ab.

Hence, generally, if, in the notation of Art. 23,

$$G = G_p \,.\, G_q \ldots G_r,$$

and G_p, G_q, ... G_r are all simple groups, the group G is monobasic.

39. Conversely, if a monobasic group G be decomposed into groups G_p, G_q, ... G_r, each of these will be a simple group. For, if the order of G be $p^\lambda \,.\, \pi$, where π does not contain the prime p, there is an element a of G of order $p^\lambda\pi$. Therefore there is also an element, namely a^π, of order p^λ; and this is the base of the group G_p which is also order p^λ. G_p is therefore a simple group.

40. Hence, if any one of the groups G_p, G_q, ... G_r is not simple, the group G is not monobasic. In fact, if G_p is the product of k simple independent groups, and no one of the groups G_q, ... G_r is the product of more than k simple groups, then G is k-basic.

41. It is an obvious corollary to the above that every group contained in a monobasic group is also monobasic.

Basicity of complete Group of Totitives. (Art. 42.)

42. If $n = p^\alpha \,.\, q^\beta \ldots r^\gamma$, where $p, q, \ldots r$ are k different odd primes, the complete group of totitives of n is k-basic. In the first place, each factor p^α, q^β, ... contributes a group to G_2, which is therefore at least k-basic. On the other hand, the group belonging to each factor

p^α, q^β, ... is monobasic (Gauss, *Disqu. Arithm.*, Art. 89); and the product of k such groups cannot be more than k-basic.

If $n = 2^\lambda . p^\alpha . q^\beta \dots r^\gamma$, where, as before, $p, q, \dots r$ are k different odd primes, the group of totitives of n is k-basic, $(k+1)$-basic, or $(k+2)$-basic according as

$$\lambda = 1, \quad = 2, \text{ or } > 2.$$

For Gauss shows (D. A., Art. 91) that the group of totitives of 2^λ is dibasic, the bases being $5, -1$. If, however, $\lambda = 2$, the group with base 5 reduces to 1; and, if $\lambda = 1$, both groups reduce to 1.

Notation of Groups.

43. It is convenient to have a notation which will indicate both the base and the order of a simple group. For this purpose I write the order as an exponent, so that a group is in fact represented by that power of its base which is congruent to unity. If a group is not simple, it is represented as a product of simple groups. Clearly, any monobasic group can be denoted similarly.

The actual value of the group symbol, considering it for the moment as a number, is $\lambda n + 1$, a unilinear function of n. Any every unilinear function of n when separated into factors, which are then replaced by their residues to modulus n, gives either a representation of a group of the totitives of n, or it expresses that the simple groups symbolised by it are not independent. It is, I think, suggestive that the group of the unilinear functions of n should be so intimately related to the groups of the totitives of n.

Example. (Arts. 44— 6.)

44. I conclude this section by applying some of the methods described to the group of totitives of 205.

The factors of 205 are 41, 5.

The totient of 41 is 40, and, since the group is monobasic, G_2, G_5 must be simple groups. Thus the group of totitives of 41 is of the form a^8, b^5. Similarly for 5, we have a simple group c^4.

Here the group for 205 is of the form

$$a^8, \; b^5, \; c^4.$$

To determine the bases, we try the orders of different totitives, and find that 2 is of order 20, which gives at once the values of b $(= 2^4)$, and c $(= 2^5)$. The element 3 is of order 8, and this may be taken for a.

Writing the groups at length, they are

$$(1, 3, 9, 27, 81, 38, 114, 137)(1, 16, 51, 201, 141)(1, 32, 204, 173).$$

45. Next, to determine the groups which are factors of G. The only group to be examined is G_2. To obtain all the elements in G_2 I form a table of the product, viz.,

	(i.)	(viii.)	(iv.)	(viii.)	(ii.)	(viii.)	(iv.)	(viii.)
(i.)	1	3	9	27	81	38	114	137
(iv.)	32	96	83	44	132	191	163	79
(ii.)	204	202	196	178	124	167	91	68
(iv.)	173	109	122	161	73	14	42	126

The Roman numerals at the top and left denote orders, the order of an element being the greater of the two orders (line-order and column-order) assigned to it.

I note that this table is not merely a table of elements, but also a table of simple groups. If, for example, we start from any element and go two places to the right, we come to element nine times the first. If thence we go one place downwards, we reach the second element $\times 32$, or the first element multiplied by 83. Hence a step, so many places to the right and so many places down, means multiplying the initial element by a certain number; which number is at once determined by starting from 1. Hence, by repeating a step until we come to the initial number, we get a group. Of course, the table may be conceived to be repeated in adjacent rectangles to allow of the steps being continued; but, equally of course, the actual repetition of the table is unnecessary. In this way we can easily form all the simple groups in G_2. To form a group of order 8, for instance, we start from 1 and make a step to any element of order 8; the continued repetition of this step gives us the elements of the group.

46. I now give the table of simple groups of G_2 as described in Art. 37. Some modifications have been made. The line of 81^2 has been divided; the groups in each line having a group of order 4 in common. The groups are not given in full, only the elements which would do for bases being named. Finally, the absolute minima are used instead of the minimum positive residues—

$$81^2, \quad \begin{array}{ll} (9, -91)^4, & (-3, -27, -38, 68)^8, \; (3, 27, 38, -68)^8, \\ (-9, 91)^4, & (96, -44, -14, -79)^8, \; (14, 79, -96, 44)^8, \end{array}$$

$$(-81)^2, \; (83, 42)^4, \quad (-83, -42)^4,$$

$$(-1)^2, \quad (32, -32)^4, \; (73, -73)^4,$$

Here there are 13 simple groups. These may be combined with G_5, giving 13 more, and G_5 may be taken by itself. Altogether, then, there are 27 monobasic groups of the totitives of 205.

There are 8 dibasic groups contained in G_2. Namely,

of order	4,	$81^2 . (-1)^2$	1
„	8,	$(-1)^2 . 9^4$, $(-1)^2 . 83^4$, $(-81)^2 . 32^4$.........	3
„	16,	$(-1)^2 . 3^8$, $(-1)^2 . 14^8$, $32^4 . 9^4$	3
„	32,	$3^8 . 32^4 (= G_2)$	1
			8

8 more dibasic groups are formed by combining G_5 with the 8 already found; so that, in all, there are 16 dibasic groups in G.

I note that a syzygy such as

$$81^2 . 83^4 = 81^2 . (-83)^4 = (-1)^2 . 83^4 = (-1)^2 (-83)^4,$$

is at once recognized on writing out *one* of the products. For example,

$$\begin{aligned}(-1)^2 . 83^4 = \;& 1, \quad 83, -81, \quad 42, \\ & -1, -83, \quad 81, -42.\end{aligned}$$

Now this is the same as $81^2 . 83^4$ (say), for it includes both the bases 81, 83, and is of the right order; it is not equivalent to $(-81)^2 . 32^4$, for it does not include the base 32.

[47. I have to thank Professor Cayley for the suggestion that "a preferable mode of treatment might be by means of the decomposition of the n^{th} roots into factors depending on the prime factors of n thus, in the example (Reuschle, p. 385, $n = 91$), writing $r = \alpha\beta$, $\alpha^7 = 1$, $\beta^{13} = 1$, the period of 24 terms is

$$\begin{array}{lll} (1, 6)(1, 8, 12, 5), & \text{read} & (\alpha+\alpha^6)(\beta+\beta^8+\beta^{12}+\beta^5) \\ +(3, 4)(6, 9, 7, 4) & & \qquad +\&\text{c. }\&\text{c.}, \\ +(2, 5)(10, 2, 3, 11), & & \end{array}$$

formed by means of 3 a prime root of 7, and 6 a prime root of 13."

The necessity of attending to the point raised by Prof. Cayley was forced upon me as soon as I began to apply the theory; and, following the lines of Arts. 35, 36, I considered in connection with any group G, of totitives of n groups derived by substituting for each element *of G its* residue with respect to a factor of n. For instance, the

group of indices of the 24-term period quoted from Reuschle is

$$(1, 27)(1, 8, 64, 57)(1, 74, 16), \quad n = 91.$$

The derived groups are

$$(1, 6)(1, 1, 1, 1)(1, 4, 2), \quad \text{mod. } 7,$$

$$(1, 1)(1, 8, 12, 5)(1, 9, 3), \quad \text{mod. } 13,$$

and the use of these is substantially equivalent to the introduction of α, β, where $\alpha^7 = 1$, $\beta^{13} = 1$.

48. I should like to state here a theorem which I had not obtained when this paper was read. The group (G) of all the totitives of n can be decomposed, in one way only, into groups, so that each factor group corresponds to one only of the prime or prime power factors of n. In fact, if $n = p^{\alpha} . P$, where p is a prime not contained in P, the group corresponding to p^{α} consists of the $p^{\alpha} - p^{\alpha-1}$ totitives of n, included in the formula

$$\lambda P + 1, \quad \lambda = 0, 1, 2, \dots p^{\alpha} - 1.$$

This group is further decomposable, and generally in one way only, into simple groups. In the exceptional case (viz., when $p = 2$, $\alpha > 2$) the group is dibasic, and the decomposition becomes determinate if we take the two bases to be $\equiv -1, 5$ (mod. 2^{α}) respectively. When G is decomposed into simple groups in this way, every group corresponds to one, and only one, of the prime factors of the totient of n

For example, $n = 91$; the group is $27^2 . 53^3 . 8^4 . 22^3$, where 22^3 means the group 22, 22^2, 22^3 ($\equiv 1$), mod. 91,

$$n = 96, \qquad G = 31^2 . 37^8 . 65^2,$$

$$n = 205, \qquad G = 42^4 . 16^5 . 96^8.$$

Jan., 1889.]

Section II. *Periods of the n^{th} roots of unity.*

1. The n^{th} roots of unity are defined by the equation

$$x^n = 1,$$

and include every power of x. If x satisfies any equation of the same form and of lowest possible degree

$$x^d = 1,$$

we shall, in analogy with the nomenclature of the preceding section (I., 6), call it a root of order d. The primitive n^{th} roots will thus be

roots of order n. It is known that, given a root, x, of order n, all the roots are expressible as powers of x; and in particular all the roots of the n^{th} order are the powers of x whose indices are totitives of n.

2. The roots of order n may be arranged in sets, which will be called *periods*, if the following conditions are satisfied:—

1. Each period consists of a sum of roots of order n.
2. In the whole system of periods each root of order n occurs once and only once.
3. Every symmetric function of the periods is a symmetric function of the roots.

The period which contains x in the first power will be called the leading period.

3. The third property of periods requires that the interchange of any pair of roots should either interchange some of the periods, or not change them at all. Suppose, then, that the leading period, which contains x, also contains x^a and x^b. If x be changed into x^a, the leading period is changed into itself (for no other period contains the root x^a). Now x^a, x^b become x^{a^2}, x^{ab} respectively. Hence, if a, b be any two of the indices of the leading period, a^2, ab are also included among these indices. It follows, therefore, that the indices of the leading period form a group of the totitives of n.

4. Next, suppose that a is a totitive not included in the indices of the leading period. Then the change of x into x^{a} changes the leading period into one of the other periods. The indices of this other period are

$$\mathrm{a},\ \mathrm{a}a,\ \mathrm{a}b,\ \ldots,$$

and, as in (I., 16), these are all different from each other and from the indices of the leading period.

5. It is an immediate consequence that every period in the system has the same number of terms; a number which, following the notation of Gauss, will be denoted by f. The number of periods, on the same authority, is called e. And it follows that

$$e \,.\, f = rn.$$

6. Conversely, when the indices of the one set of roots form a group, and the indices of the other sets are derived from those of the leading set by multiplication as in (I., 16), these sets are periods. It is only needful to prove that the sets have the third property of periods, for the other two are already assured. Now, if we change any

primitive root x^a into another, x^b, the change is equivalent to replacing x by x^c, where c is uniquely determined by the congruence

$$ac \equiv b, \text{ mod. } n.$$

This is equivalent to multiplying every index by the totitive c, and, as remarked in (I., 17), the system of a group and its multiples is not altered. The interchange, therefore, of any two of the roots does not alter the value of any symmetric function of the sets of roots. These sets are therefore periods.

7. If n is a prime, the totitives of n form a monobasic group; so that the indices of the leading period, being a group of the totitives of n, also form a monobasic group. Also, when f is prime, the indices of the leading period form a group which is necessarily monobasic, since its order is prime.

8. If the indices of a leading period be multiplied by a number which is not prime to n, we obtain the indices of what may be called an imprimitive period. The number of different roots in an imprimitive period is a factor of f (including f itself and 1 as extreme cases), so that each root, in effect, is multiplied by an integer, the same for all.

9. The fundamental properties of periods are—

(1) They are roots of an equation, in which the coefficient of the highest power of η is unity and all the other coefficients are integers.

(2) This equation is irreducible.

(3) The product of any two periods, including as a particular case the square of any period, is expressible as a linear function with integral coefficients of the primitive and imprimitive periods.

10. We proceed to show that these properties are true of periods as above defined. The proofs are precisely the same as the proofs given in the text books (for instance, Bachmann's *Kreistheilung*, which I follow) for the special case of monobasic periods.

11. The proof of the first property depends on the fact that the primitive roots themselves satisfy an equation of the kind specified Hence every rational integral symmetric function of these roots is an integer; therefore every such symmetric function of the periods is an integer; and therefore the periods are roots of an equation the coefficient of the highest term in which is unity, while all the other coefficients are integers.

12. The proof that the equation in question is irreducible is likewise based on the fact that the equation $Fx = 0$, whose roots are the primitive roots of unity, is irreducible. For, let $f\eta = 0$ be the equation satisfied by the periods, and suppose that $f\eta$ contains a rational factor $g\eta$. The equation $g\eta = 0$ must therefore be satisfied by some of the periods η. Replacing η by its expression in terms of the roots, the equation $g\eta$ becomes an equation in x, which has one, and therefore all its roots in common with the irreducible equation $Fx = 0$. Hence $g\eta$ is not affected by any transposition of the roots x, and therefore it is not changed when any period of the set considered is replaced by any other period. Thus, *either* $g\eta = 0$ must be an equation satisfied by all the periods or by none of the periods, which is tantamount to saying that $f\eta$ is irreducible; *or* some at least of the periods must be equal. In the latter case it can be shown that all the periods must vanish; so that the cyclotomic function $f\eta$ reduces to a single term η^e.

13. To prove the third property, consider the product of the two periods whose indices are

$$1,\ a,\ b \ldots c,$$

$$\mathrm{a},\ \mathrm{a}a,\ \mathrm{a}b,\ \ldots,\ \mathrm{a}c,$$

respectively. This product is

$$\eta_1\eta_{\mathrm{a}} = (x+x^a+x^b+\ldots+x^c)(x^{\mathrm{a}}+x^{\mathrm{a}a}+x^{\mathrm{a}b}+\ldots+x^{\mathrm{a}c})$$

and consists of f^2 terms. It is to be shown that these make up $f.f$-nomial periods included in the set derived from η_1. In fact, if we multiply any one term, say x, of the period η_1 into all the terms of the period η_{a}, we shall obtain one term for each of the f periods in the product. For example, we have

$$x \,.\, x^{\mathrm{a}a} = x^{1+\mathrm{a}a},$$

and I say that the period whose indices are.

$$1+\mathrm{a}a,\quad a\,(1+\mathrm{a}a),\quad b\,(1+\mathrm{a}a) \ldots c\,(1+\mathrm{a}a)$$

is contained in the product $\eta_1\ \eta_{\mathrm{a}}$. For

$$a,\quad a^2,\quad ab, \ldots ac$$

are contained in the group of indices $(1, a, b, \ldots c)$,

therefore $$\mathrm{a}a,\ \mathrm{a}a^2,\quad \mathrm{a}ab, \ldots \mathrm{a}ac$$

are contained in the indices of the period η_{a}.

Hence we have

$$\eta_1 \eta_a = \eta_{1+a} + \eta_{1+aa} + \eta_{1+ab} + \ldots + r_{1+ac}$$

These periods are not always primitive periods nor all different.

Historical Note.

14. Gauss, in his *Disquisitiones Arithmeticae*, Arts. 46, 343, starts with the consideration of a geometric series, and is therefore led to monobasic periods. These are indeed the only periods that can occur in the cases considered by Gauss, viz., when n is a prime number. But when the case of n, a composite number, was investigated by Kummer, the monobasic period seems to have been regarded as the only period. For instance, in Reuschle's *Tafeln* (1875), which are based upon Kummer's theory, all periods except monobasic periods are expressly excluded. For example, on page 385, there is an "Anmerkung zu $n = 91$. Es gibt hier Summen von 24 Wurzeln welche eine eigenthümliche Bedeutung haben, ohne Perioden zu sein, und der wirklichen Berechnung der Congruenzwurzeln hin und wieder zu Grunde liegen, nämlich," . . . . and then he gives the periods derived from the dibasic groups $27^2 . 8^4 . 16^3$ and $27^2 . 8^4 . 9^3$, which he gives at length. He also gives the corresponding cyclotomic functions

$$\eta^3 - \eta^2 - 30\eta + 64, \quad \eta^3 - \eta^2 - 30\eta - 27.$$

He alludes to such aggregates of roots in connection with

$$n = 24,\ 40,\ 48,\ 56,\ 60,\ 84,\ 88,\ 91,\ 105;$$

but the enumeration is not, and probably was not meant to be complete.

15. The only reference I have found to periods other than monobasic periods is in an Excursus of Prof. Sylvester "On the Divisors of Cyclotomic Functions," (*American Journal of Mathematics*, vol. ii.). He gives some periods which are not monobasic, for instance (p. 378) the period whose indices are 1, 8, 13, 20 for $n = 21$. And on page 380 (top) he expressly draws attention to the fact that the indices of his period need not necessarily include any primitive root. But the theory of the periods is not given, and I have met with no further statement on the subject. It was the attempt to reconcile Professor Sylvester's remark with the usual theory of periods that led to the research now communicated.

Thursday, December 13th, 1888.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Messrs. C. W. C. Barlow, B.A., Scholar of St. Peter's College, Cambridge; H. F. Baker, B.A., Fellow of St. John's College, Cambridge; W. N. Roseveare, B.A., Fellow of St. John's College, Cambridge; Miss M. T. Meyer, Mathematical Lecturer at Girton College, Cambridge, and Major D. O'Callaghan, R.A., Royal Arsenal, Woolwich, were elected members.

The following communications were made:—

A Geometrical Note: by H. M. Taylor, M.A. (communicated by Dr. Glaisher, F.R.S.).

The Equilibrium of a Thin Elastic Bowl: A. E. H. Love, M.A.

A Method of Transformation with the Aid of Congruences of a Particular Type: J. Brill, M.A.

Notes and Illustrations on a former paper "On a Method in the Analysis of Ternary Forms:" the President (Prof. Greenhill, F.R.S., in the Chair).

The following presents were received:—

"Educational Times," for December.

"The Nautical Almanack" for 1892; from the Lords Commissioners of the Admiralty.

"A Treatise on Hydrodynamics," by A. B. Basset, Vol. II., 8vo; Cambridge, 1888.

"The Elementary Geometry of Conics," by C. Taylor, D.D., 5th edition, 8vo; Cambridge, 1888.

"Proceedings of the Canadian Institute," Third Series, Vol. VI., Fasc. No. 1.

"Bulletin des Sciences Mathématiques," Tome XII.; September and October, 1888.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 69 and 70; Firenze, 1888.

"Acta Mathematica," XII., 1.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. IV., Fasc. 12, and 13; Vol. VI., Fasc. 3, 4, 5 (all) 1888; Roma.

"Journal für die reine und angewandte Mathematik," Band 104, Heft. I.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXIII., Livr. 1; Harlem, 1888.

"Jahrbuch über die Fortschritte der Mathematik," Band XVIII., Heft 1, Jahrgang 1886; Berlin, 1888.

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo II., No. 4; Mexico, 1888.

"Prace Matematyczno-Fizyczne," Tom. I.; Warszawa.

On the Equilibrium of a Thin Elastic Spherical Bowl.

By A. E. H. Love, B.A.

[*Read Dec.* 13*th*, 1888.]

1. In a recent paper (*Phil. Trans.*, 1888) I have considered the deformation of a thin elastic shell, and have obtained the general equations of motion and equilibrium, under any system of applied forces and edge-tractions, subject to the condition that the displacement of any point of the shell is always small. In the present communication, the theory there developed is applied to some cases of the equilibrium of a spherical bowl.*

In the paper referred to, it was shown that the potential energy of deformation of the shell consisted of two terms, one depending on functions σ_1 σ_2, ϖ defining the stretching of the middle-surface, and the other depending on functions κ_2, λ_1, κ_1 defining the bending of the middle-surface. Of these the first is proportional to the thickness of the shell, and the second is proportional to the cube of the thickness. It was shown to be inadmissible to suppose the middle-surface unstretched, because the boundary conditions cannot then be satisfied; and it then appeared that, in case the boundary conditions can be satisfied, it is legitimate to neglect the term of the potential energy depending on the bending as unimportant compared with the term depending on the stretching. It is only for certain distributions of bodily force and edge-traction that the boundary conditions can be satisfied. These will be the cases here treated. I may remark that the problems solved are of comparatively little physical interest, but I think the differential equations whose solution is obtained justify me in bringing the results before the society.

The bodily forces acting on any line-element of the shell, which is normal to its middle-surface, can be reduced to a force and a couple at the point in which the element meets the middle-surface. The

* In Lord Rayleigh's paper on the "Bending of Surfaces of Revolution" (*Proceedings*, Vol. XIII.), a different theory of the behaviour of a strained elastic shell is advanced. Lord Rayleigh has also extended his method to the case of cylindrical shells, in a paper read before the Royal Society, in December, 1888. I have discussed Lord Rayleigh's method of procedure in my paper on the "Small Free Vibrations and Deformation of a Thin Elastic Shell," in the *Phil. Trans.*, 1888. I do not regard the question as yet settled, nor do I think the present occasion appropriate for its discussion.

components of the force along the lines of curvature and the normal are taken to be X, Y, Z, the components of the couple about the lines of curvature are taken to be L, M. These are estimated per unit of area of the middle-surface.

In like manner, the edge-tractions can be reduced to a force whose components along the lines of curvature and the normal are A, B, C, and a couple whose components about the lines of curvature are U, V. These are estimated per unit of length of the curve in which the middle-surface cuts the edge.

There is no couple about the normal, because all the forces compounded meet it.

It appears from the boundary conditions given in the paper referred to, viz., equations (33), (34), (35), on pp. 519 and 520, that the last two of these contain only terms depending on the bending, and on the force- and couple-components C, U, V, L, M; and we may therefore neglect the terms depending on the bending, and form approximate equations of equilibrium depending on the stretching only, if the quantities C, U, V, L, M all vanish.

This is the case when the bodily-forces and edge-tractions, acting on a line of the shell drawn normal to its middle-surface, have no moments about any line in the middle-surface, and when there is no edge-traction along the normal to the middle-surface.

2. In the equations obtained in the paper referred to, the displacement of a point on the middle-surface is estimated by its components along the lines of curvature and the normal. We suppose the lines of curvature to be drawn, and to be given by parameters α, β; we further suppose a system of orthogonal surfaces constructed of which the middle-surface is one, and the lines of curvature are its intersections with the other two co-orthogonal families of surfaces. The parameters of the three families of surfaces are α, β, γ, and $\gamma =$ const. is the equation of the middle-surface. Writing

$$h_1^2 = \left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\alpha}{\partial y}\right)^2 + \left(\frac{\partial\alpha}{\partial z}\right)^2, \qquad h_2^2 = \left(\frac{\partial\beta}{\partial x}\right)^2 + \left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2,$$

$$h_3^2 = \left(\frac{\partial\gamma}{\partial x}\right)^2 + \left(\frac{\partial\gamma}{\partial y}\right)^2 + \left(\frac{\partial\gamma}{\partial z}\right)^2,$$

the element of length is

$$(d\alpha/h_1)^2 + (d\beta/h_2)^2 + (d\gamma/h_3)^2.$$

The principal radii of curvature of the normal sections through $d\alpha$,

and $d\beta$ are ρ_1, ρ_2, where

$$\frac{1}{\rho_1} = h_1 h_3 \frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right), \qquad \frac{1}{\rho_2} = h_2 h_3 \frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\right).$$

The displacement of any point of the middle-surface is taken to be u along $\beta =$ const., v along $\alpha =$ const., w along the normal outwards.

The extensions of the line-elements initially lying along the lines of curvature are σ_1, σ_2, where

$$\left.\begin{aligned} \sigma_1 &= h_1 \frac{\partial u}{\partial\alpha} + h_1 h_2 v \frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + \frac{w}{\rho_1} \\ \sigma_2 &= h_2 \frac{\partial v}{\partial\beta} + h_1 h_2 u \frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right) + \frac{w}{\rho_2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and the shear of these two line-elements is ϖ, where

$$\varpi = \frac{h_1}{h_2}\frac{\partial}{\partial\alpha}(h_2 v) + \frac{h_2}{h_1}\frac{\partial}{\partial\beta}(h_1 u) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equations of equilibrium become, by the omission of the couples L, M, and of the terms depending on the bending,

$$\left.\begin{aligned} &-\frac{X}{h_1 h_2} + 2nh\Big[-2\frac{\partial}{\partial\alpha}\Big\{\frac{1}{h_2}\Big(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\Big)\Big\} \\ &\qquad + 2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\Big(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\Big) - h_1\frac{\partial}{\partial\beta}\left(\frac{\varpi}{h_1^2}\right)\Big] = 0 \\ &-\frac{Y}{h_1 h_2} + 2nh\Big[-2\frac{\partial}{\partial\beta}\Big\{\frac{1}{h_1}\ \frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\Big)\Big\} \\ &\qquad + 2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\Big(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\Big) - h_2\frac{\partial}{\partial\alpha}\left(\frac{\varpi}{h_2^2}\right)\Big] = 0 \\ &-\frac{Z}{h_1 h_2} + 2nh\Big[\frac{1}{\rho_1}\Big(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\Big) \\ &\qquad\qquad + \frac{1}{\rho_2}\Big(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\Big)\Big]\frac{2}{h_1 h_2} = 0 \end{aligned}\right\} \ldots\ldots(3),$$

where $2h$ is the thickness of the shell, and m, n are constants of elasticity, viz., n is the rigidity and $m = k + \frac{1}{3}n$, where k is the resistance to compression.

The boundary conditions become

$$\left.\begin{aligned}-A+2nh\left[2\lambda\left(\frac{2m}{m+n}\sigma_1+\frac{m-n}{m+n}\sigma_2\right)+\mu\varpi\right]&=0\\ -B+2nh\left[2\mu\left(\frac{2m}{m+n}\sigma_2+\frac{m-n}{m+n}\sigma_1\right)+\lambda\varpi\right]&=0\end{aligned}\right\}\dots\dots\dots(4),$$

where λ and μ are the cosines of the angles which the normal to the edge, drawn on the middle-surface and outwards from the edge, makes with the lines of curvature $\beta=$ const. and $\alpha=$ const. at the edge.

3. I propose to apply these equations to the equilibrium of a thin spherical bowl bounded by a small circle. The poles of the small circle define a system of meridians $\theta=$ const., and the parallel small circles a system of parallels $\phi=$ const., and these are lines of curvature, so that we may take

$$\alpha=\theta,\quad \beta=\phi,\quad \gamma=r,$$

where r is the radius of the sphere concentric with the middle-surface and passing through any point, and $r=a$ is the equation of the middle-surface. The values of λ, μ at the edge are $\lambda=1$, $\mu=0$.

In this case, we have

$$\left.\begin{aligned}1/h_1=r,\quad 1/h_2=r\sin\theta,\quad 1/h_3=1\\ \rho_1=a,\qquad \rho_2=a\end{aligned}\right\}\dots\dots\dots\dots\dots(5),$$

$$\left.\begin{aligned}\sigma_1&=\frac{1}{a}\frac{\partial u}{\partial\theta}+\frac{w}{a}\\ \sigma_2&=\frac{1}{a\sin\theta}\frac{\partial v}{\partial\phi}+\frac{u}{a}\cot\theta+\frac{w}{a}\\ \varpi&=\frac{1}{a}\frac{\partial v}{\partial\theta}+\frac{1}{a\sin\theta}\frac{\partial u}{\partial\phi}-\frac{v}{a}\cot\theta\end{aligned}\right\}\dots\dots\dots\dots\dots(6);$$

and, if we write for shortness,

$$a^2X/2nh=X',\quad a^2Y/2nh=Y',\quad a^2Z/2nh=Z'\dots\dots\dots(7),$$

equations (3) become

$$X'+\frac{\partial^2u}{\partial\theta^2}+\cot\theta\frac{\partial u}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}+u\,(1-\cot^2\theta)-\frac{2\cos\theta}{\sin^2\theta}\frac{\partial v}{\partial\phi}$$
$$+\frac{3m-n}{m+n}\frac{\partial}{\partial\theta}\left[\frac{\partial u}{\partial\theta}+u\cot\theta+\frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}+2w\right]=0\dots\dots(8),$$

$$Y'+\frac{\partial^2 v}{\partial\theta^2}+\cot\theta\frac{\partial v}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}+v\left(1-\cot^2\theta\right)+\frac{2\cos\theta}{\sin^2\theta}\frac{\partial u}{\partial\phi}$$

$$+\frac{3m-n}{m+n}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\partial u}{\partial\theta}+u\cot\theta+\frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}+2w\right]=0 \quad \text{......(9)},$$

$$Z'-2\frac{3m-n}{m+n}\left[\frac{\partial u}{\partial\theta}+u\cot\theta+\frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}+2w\right]=0 \quad \text{..............(10)}.$$

Hence u and v must be found from the equations

$$\left.\begin{aligned}&\frac{\partial^2 u}{\partial\theta^2}+\cot\theta\frac{\partial u}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}+u\left(1-\cot^2\theta\right)-\frac{2\cos\theta}{\sin^2\theta}\frac{\partial v}{\partial\phi}\\ &\qquad=-\left(X'+\frac{1}{2}\frac{\partial Z'}{\partial\theta}\right)\\ &\frac{\partial^2 v}{\partial\theta^2}+\cot\theta\frac{\partial v}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}+v\left(1-\cot^2\theta\right)+\frac{2\cos\theta}{\sin^2\theta}\frac{\partial u}{\partial\phi}\\ &\qquad=-\left(Y'+\frac{1}{2}\frac{1}{\sin\theta}\frac{\partial Z'}{\partial\phi}\right)\end{aligned}\right\}\text{...(11)},$$

and then w is determined by (10).

4. To solve these equations, we suppose $u\propto\cos s\phi$, $v\propto\sin s\phi$, where s is an integer, then (11) become

$$\left.\begin{aligned}&\frac{\partial^2 u}{\partial\theta^2}+\cot\theta\frac{\partial u}{\partial\theta}+u\left\{2-(1+s^2)\operatorname{cosec}^2\theta\right\}-2sv\cot\theta.\operatorname{cosec}\theta\\ &\qquad=-\left(X'+\tfrac{1}{2}\frac{\partial Z'}{d\theta}\right)\\ &\frac{\partial^2 v}{\partial\theta^2}+\cot\theta\frac{\partial v}{\partial\theta}+v\left\{2-(1+s^2)\operatorname{cosec}^2\theta\right\}-2su\cot\theta\operatorname{cosec}\theta\\ &\qquad=-\left(Y'+\tfrac{1}{2}\operatorname{cosec}\theta\frac{\partial Z'}{\partial\phi}\right)\end{aligned}\right\}\text{...(12)}.$$

Putting $u+v=y_1$, $u-v=y_2$,

$$\left.\left(X'+\tfrac{1}{2}\frac{\partial Z'}{\partial\theta}\right)=-\tfrac{1}{2}\left(Y_1+Y_2\right)\quad\left(Y'+\tfrac{1}{2}\operatorname{cosec}\theta\frac{\partial Z'}{\partial\phi}\right)=-\tfrac{1}{2}\left(Y_1-Y_2\right)\right\}$$

$$\text{............ (13)};$$

these become

$$\left.\begin{aligned}&\frac{d^2 y_1}{d\theta^2}+\cot\theta\frac{dy_1}{d\theta}+y_1\left[2-(1+s^2)\operatorname{cosec}^2\theta\right]-2s\operatorname{cosec}\theta\cot\theta\, y_1=Y_1\\ &\frac{d^2 y_2}{d\theta^2}+\cot\theta\frac{dy_2}{d\theta}+y_2\left[2-(1+s^2)\operatorname{cosec}^2\theta\right]+2s\operatorname{cosec}\theta\cot\theta\, y_2=Y_2\end{aligned}\right\}$$

$$\text{............ (14)}.$$

5. We can solve these equations completely when we know a particular integral of each of the equations derived from them, by making Y_1, Y_2 zero.

Now, from the way in which the equations were formed, it is plain that one particular solution of the system will be derived by putting $\sigma_1, \sigma_2, \varpi = 0$. Hence, to find a particular solution for u and v, we have

$$\left.\begin{aligned} \frac{\partial u}{\partial \theta}+w &= 0 \\ \frac{1}{\sin\theta}\frac{\partial v}{\partial \phi}+w+u\cot\theta &= 0 \\ \frac{1}{\sin\theta}\frac{\partial u}{\partial \phi}+\frac{\partial v}{\partial \theta}-v\cot\theta &= 0 \end{aligned}\right\} \quad \text{.................. (15)},$$

the equations of inextensibility.

From these
$$\left.\begin{aligned} \frac{\partial}{\partial\phi}\left(\frac{u}{\sin\theta}\right)+\sin\theta\frac{\partial}{\partial\theta}\left(\frac{v}{\sin\theta}\right) &= 0 \\ \frac{\partial}{\partial\phi}\left(\frac{v}{\sin\theta}\right)-\sin\theta\frac{\partial}{\partial\theta}\left(\frac{u}{\sin\theta}\right) &= 0 \end{aligned}\right\} \quad \text{............ (16)},$$

so that $u\operatorname{cosec}\theta$ and $v\operatorname{cosec}\theta$ are conjugate solutions of the equation

$$\frac{\partial^2 X}{\partial\phi^2}+\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial X}{\partial\theta}\right)=0 \quad \text{.............. (17)};$$

hence particular solutions are

$$u=\sin\theta\tan^s\tfrac{1}{2}\theta\cos s\phi, \quad u=\sin\theta\cot^s\tfrac{1}{2}\theta\cos s\phi,^*$$
$$v=\sin\theta\tan^s\tfrac{1}{2}\theta\sin s\phi, \quad v=\sin\theta\cot^s\tfrac{1}{2}\theta\sin s\phi.$$

We may show that $\sin\theta\tan^s\frac{1}{2}\theta$ is a particular integral of the equation for y_1 when $Y_1=0$, and $\sin\theta\cot^s\frac{1}{2}\theta$ is a particular integral of the equation for y_2 when $Y_2=0$.

Writing
$$\left.\begin{aligned} u_0 &= \sin\theta\tan^s\tfrac{1}{2}\theta \\ v_0 &= \sin\theta\cot^s\tfrac{1}{2}\theta \end{aligned}\right\} \quad \text{.................. (18)},$$
and

we have, by equation (17), for u_0 and v_0,

$$\sin\theta\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\left(\frac{u_0}{\sin\theta}\right)\right\}-\frac{s^2u_0}{\sin\theta}=0.$$

Hence
$$\frac{d^2u_0}{d\theta^2}+u_0=\cot\theta\frac{du_0}{d\theta}+u_0\left(s^2\operatorname{cosec}^2\theta-\cot^2\theta\right),$$

* *Cf.* Lord Rayleigh "On the Infinitesimal Bending of Surfaces of Revolution," *Proceedings*, Vol. XIII.

also $\dfrac{du_0}{d\theta} = u_0 \cot\theta + \tfrac{1}{2} s u_0 \sec^2 \tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta = u_0 (s + \cos\theta) \operatorname{cosec}\theta$;

and $$\frac{d^2 v_0}{d\theta^2} + v_0 = \cot\theta \frac{dv_0}{d\theta} + v_0 (s^2 \operatorname{cosec}^2\theta - \cot^2\theta),$$

also $\dfrac{dv_0}{d\theta} = v_0 \cot\theta - \tfrac{1}{2} s v_0 \operatorname{cosec}^2 \tfrac{1}{2}\theta \tan\tfrac{1}{2}\theta = v_0 (\cos\theta - s) \operatorname{cosec}\theta$;

thus $$\frac{d^2 u_0}{d\theta^2} + \cot\theta \frac{du_0}{d\theta}$$

$$= u_0 \left[-1 - \cot^2\theta + s^2 \operatorname{cosec}^2\theta + 2\cot^2\theta + 2s \cot\theta \operatorname{cosec}\theta \right],$$

and $$\frac{d^2 v_0}{d\theta^2} + \cot\theta \frac{dv_0}{d\theta}$$

$$= v_0 \left[-1 - \cot^2\theta + s^2 \operatorname{cosec}^2\theta + 2\cot^2\theta - 2s \cot\theta \operatorname{cosec}\theta \right],$$

coinciding with (14).

We can hence deduce the general solutions for y_1, y_2.

6. Let $y_1 = y u_0$.

Then $$y \left\{ \frac{d^2 u_0}{d\theta^2} + \cot\theta \frac{du_0}{d\theta} + \left[2 - (1+s^2)\operatorname{cosec}^2\theta - \frac{2s\cos\theta}{\sin^2\theta} \right] u_0 \right\}$$

$$+ u_0 \frac{d^2 y}{d\theta^2} + 2 \frac{du_0}{d\theta}\frac{dy}{d\theta} + \cot\theta \frac{dy}{d\theta} u_0 = Y_1.$$

Writing y for $\dfrac{dy}{d\theta}$, and multiplying by u_0, we have

$$\frac{u_0^2}{\sin\theta} \frac{d}{d\theta} (\sin\theta\, y') + \frac{1}{\sin\theta} (2 \sin\theta\, y' u_0) \frac{du_0}{d\theta} = Y_1 u_0.$$

Put $\cos\theta = \mu$, then this is

$$\frac{d}{d\mu} (u_0^2 \sin\theta\, y') = -Y_1 u_0,$$

so that $$u_0^2 \sin^2\theta\, y' = \int Y_1 u_0 \sin\theta\, d\theta + B_1'.$$

Hence integrating,

$$y_1 = u_0 \left[A_1 + B_1' \int \frac{d\theta}{u_0^2 \sin\theta} + \int \left\{ \frac{1}{u_0^2 \sin\theta} \int Y_1 u_0 \sin\theta\, d\theta \right\} d\theta \right] \text{(19)}.$$

In like manner, writing $y_2 = z v_0$, we find

$$y_2 = v_0 \left[A_2 + B_2' \int \frac{d\theta}{v_0^2 \sin\theta} + \int \left\{ \frac{1}{v_0^2 \sin\theta} \int Y_2 v_0 \sin\theta\, d\theta \right\} d\theta \right] \ldots \text{(20)}.$$

In these solutions A_1, A_2, B_1', B_2' are arbitrary constants.

Now to find the integrals

$$\int u_0^{-2} \operatorname{cosec} \theta \, d\theta, \quad \int v_0^{-2} \operatorname{cosec} \theta \, d\theta,$$

put
$$\log \tan \tfrac{1}{2}\theta = z,$$

then
$$\int \cot^{2s} \tfrac{1}{2}\theta \operatorname{cosec}^3 \theta \, d\theta = -\int \operatorname{cosec}^4 \theta \cot^{2s} \tfrac{1}{2}\theta \, d(\cos\theta)$$

$$= \int e^{-2sz} \cosh^4 z \, d(\tanh z)$$

$$= \int e^{-2sz} \cosh^2 z \, dz$$

$$= \tfrac{1}{4} \int \left(2e^{-2sz} + e^{-2(s+1)z} + e^{-2(s-1)z}\right) dz$$

$$= -\tfrac{1}{8} \cot^{2s} \tfrac{1}{2}\theta \left[\frac{2}{s} + \frac{1}{s+1} \cot^2 \tfrac{1}{2}\theta + \frac{1}{s-1} \tan^2 \tfrac{1}{2}\theta\right] \quad \text{..............(21).}$$

In like manner

$$\int \tan^{2s} \tfrac{1}{2}\theta \operatorname{cosec}^3 \theta \, d\theta = \tfrac{1}{8} \tan^{2s} \tfrac{1}{2}\theta \left[\frac{2}{s} + \frac{1}{s+1} \tan^2 \tfrac{1}{2}\theta + \frac{1}{s-1} \cot^2 \tfrac{1}{2}\theta\right] \quad \text{..............(22).}$$

Thus

$$y_1 = A_1 \sin\theta \tan^s \tfrac{1}{2}\theta + B_1 \sin\theta \cot^s \tfrac{1}{2}\theta \left[\frac{2}{s} + \frac{1}{s+1} \cot^2 \tfrac{1}{2}\theta + \frac{1}{s-1} \tan^2 \tfrac{1}{2}\theta\right]$$
$$+ \sin\theta \tan^s \tfrac{1}{2}\theta \int \left\{\operatorname{cosec}^3 \theta \cot^{2s} \tfrac{1}{2}\theta \int (Y_1 \sin^2\theta \tan^s \tfrac{1}{2}\theta) \, d\theta\right\} d\theta \quad \text{... (22),}$$

$$y_2 = A_2 \sin\theta \cot^s \tfrac{1}{2}\theta + B_2 \sin\theta \tan^s \tfrac{1}{2}\theta \left[\frac{2}{s} + \frac{1}{s+1} \tan^2 \tfrac{1}{2}\theta + \frac{1}{s-1} \cot^2 \tfrac{1}{2}\theta\right]$$
$$+ \sin\theta \cot^s \tfrac{1}{2}\theta \int \left\{\operatorname{cosec}^3 \theta \tan^{2s} \tfrac{1}{2}\theta \int (Y_2 \sin^2\theta \cot^s \tfrac{1}{2}\theta) \, d\theta\right\} d\theta \quad \text{......(23).}$$

This is the complete solution of the system of differential equations (14) in the general case.

7. When $s = 0$, or $s = 1$, we get failing cases.

(i.) When $s = 0$, the equations for y_1, y_2 are

$$\left.\begin{aligned} \frac{d^2 y_1}{d\theta^2} + \cot\theta \frac{dy_1}{d\theta} + (2 - \operatorname{cosec}^2\theta) \, y_1 &= Y_1 \\ \frac{d^2 y_2}{d\theta^2} + \cot\theta \frac{dy_2}{d\theta} + (2 - \operatorname{cosec}^2\theta) \, y_2 &= Y_2 \end{aligned}\right\} \quad \text{........... (24),}$$

and $\sin\theta$ is a particular integral when Y_1, Y_2 are zero.

Thus, proceeding as before, we obtain

$$y_1 = \sin\theta \left[A_1 + B_1' \int \operatorname{cosec}^3 \theta \, d\theta + \int \left\{\operatorname{cosec}^3 \theta \int Y_1 \sin^2 \theta \, d\theta\right\} d\theta\right],$$

and a similar expression for y_2.

Hence

$$\left.\begin{aligned} y_1&=A_1\sin\theta+B_1(\cot\theta-\sin\theta\log\tan\tfrac{1}{2}\theta)+\sin\theta\int\{\operatorname{cosec}^3\theta\int Y_1\sin^2\theta\,d\theta\}\,d\theta\\ y_2&=A_2\sin\theta+B_2(\cot\theta-\sin\theta\log\tan\tfrac{1}{2}\theta)+\sin\theta\int\{\operatorname{cosec}^3\theta\int Y_2\sin^2\theta\,d\theta\}\,d\theta\end{aligned}\right\}\quad\text{..............(25).}$$

In this case we could find u, v immediately from (12), putting $\frac{\partial Z'}{\partial\phi}=0$ on account of the symmetry.

Thus, or by (25), we obtain

$$\left.\begin{aligned} u&=A_1\sin\theta+B_1(\cot\theta-\sin\theta\log\tan\tfrac{1}{2}\theta)\\ &\qquad+\sin\theta\int\left\{\operatorname{cosec}^3\theta\int-\sin^2\theta\left(X'+\tfrac{1}{2}\frac{\partial Z'}{\partial\theta}\right)d\theta\right\}d\theta\\ v&=A_2\sin\theta+B_2(\cot\theta-\sin\theta\log\tan\tfrac{1}{2}\theta)\\ &\qquad+\sin\theta\int\left\{\operatorname{cosec}^3\theta\int-\sin^2\theta\,Y'd\theta\right\}d\theta\end{aligned}\right\}\text{...(26).}$$

(ii.) When $s=1$, we write μ for $\cos\theta$, and the equations become

$$\left.\begin{aligned} \frac{d}{d\mu}\left[(1-\mu^2)\frac{dy_1}{d\mu}\right]+2y_1\left(1-\frac{1+\mu}{1-\mu^2}\right)&=Y_1\\ \frac{d}{d\mu}\left[(1-\mu^2)\frac{dy_2}{d\mu}\right]+2y_2\left(1-\frac{1-\mu}{1-\mu^2}\right)&=Y_2\end{aligned}\right\}\quad\text{..............(27).}$$

An integral of the first is

$$y_1=1-\mu=1-\cos\theta=2\sin^2\tfrac{1}{2}\theta,\ \text{when}\ Y_1=0,$$

so $$y_2=1+\mu=1+\cos\theta=2\cos^2\tfrac{1}{2}\theta,\ \text{when}\ Y_2=0.$$

Hence the complete primitives

$$\begin{aligned} y_1&=2\sin^2\tfrac{1}{2}\theta\left[A_1'+B_1'\int\operatorname{cosec}\theta\operatorname{cosec}^4\tfrac{1}{2}\theta\,d\theta\right]\\ &\qquad+2\sin^2\tfrac{1}{2}\theta\int\left\{\operatorname{cosec}\theta\operatorname{cosec}^4\tfrac{1}{2}\theta\int Y_1\sin\theta\sin^2\tfrac{1}{2}\theta\,d\theta\right\}d\theta,\\ y_2&=2\cos^2\tfrac{1}{2}\theta\left[A_2'+B_2'\int\operatorname{cosec}\theta\sec^4\tfrac{1}{2}\theta\,d\theta\right]\\ &\qquad+2\cos^2\tfrac{1}{2}\theta\int\left\{\operatorname{cosec}\theta\sec^4\tfrac{1}{2}\theta\int Y_2\sin\theta\cos^2\tfrac{1}{2}\theta\,d\theta\right\}d\theta.\end{aligned}$$

Observing that

$$\begin{aligned}\int\operatorname{cosec}\theta\operatorname{cosec}^4\tfrac{1}{2}\theta\,d\theta&=\log\tan\tfrac{1}{2}\theta-\frac{2-\cos\theta}{(1-\cos\theta)^2},\\ \int\operatorname{cosec}\theta\sec^4\tfrac{1}{2}\theta\,d\theta&=\log\tan\tfrac{1}{2}\theta+\frac{2+\cos\theta}{(1+\cos\theta)^2},\end{aligned}$$

we see that these may be written

$$y_1 = A_1(1-\cos\theta)+B_1\left[2\sin^2\tfrac{1}{2}\theta\log\tan\tfrac{1}{2}\theta-\frac{2-\cos\theta}{1-\cos\theta}\right]$$

$$+2\sin^2\tfrac{1}{2}\theta\int\left\{\operatorname{cosec}\theta\operatorname{cosec}^4\tfrac{1}{2}\theta\int Y_1\sin\theta\sin^2\tfrac{1}{2}\theta\,d\theta\right\}d\theta \quad\ldots\ldots\ldots(28),$$

$$y_2 = A_2(1+\cos\theta)+B_2\left[2\cos^2\tfrac{1}{2}\theta\log\tan\tfrac{1}{2}\theta+\frac{2+\cos\theta}{1+\cos\theta}\right]$$

$$+2\cos^2\tfrac{1}{2}\theta\int\left\{\operatorname{cosec}\theta\sec^4\tfrac{1}{2}\theta\int Y_2\sin\theta\cos^2\tfrac{1}{2}\theta\,d\theta\right\}d\theta \quad\ldots\ldots\ldots(29).$$

By equations (22) and (23), (25), and (28) and (29), we have y_1, y_2 in all cases; u and v are then to be found from

$$u = \tfrac{1}{2}(y_1+y_2), \quad v = \tfrac{1}{2}(y_1-y_2),$$

and w is given by equation (10).

To satisfy the boundary conditions, we shall require to know $\sigma_1, \sigma_2, \varpi$. These are to be calculated from the values found for u, v, w by means of equations (6).

On substituting in the boundary conditions (4), we shall be able to determine the arbitrary constants.

8. We proceed to consider some examples.

Example I.—A spherical bowl, bounded by the horizontal plane $\theta = \alpha$, is acted on by a normal pressure on its middle-surface everywhere proportional to the depth below the bounding plane, and is supported by forces applied to the edge in the directions of the tangents to the meridians on the middle-surface: it is required to find the displacement.

This is the case of a bowl filled with liquid, since the state of strain in an element of the bowl, produced by surface tractions applied to its curved surface, is the same as when corresponding bodily forces are applied to its middle-surface.*

In this problem

$$X = 0, \quad Y = 0, \quad Z' = C_1(\cos\theta-\cos\alpha),$$

where $$C_1 = g\rho' a^3/2nh$$

in the case of a fluid of density ρ'.

* The method by which M. Boussinesq has proved this result for plates holds equally for thin shells, see Liouville's *Journal de Math.*, 1871.

Since all the conditions are symmetrical with respect to the axis, v is zero, and u and w are independent of ϕ, and the proper solution is (25). Hence, observing that

$$\int \left\{\operatorname{cosec}^3\theta \int \sin^3\theta\, d\theta\right\} d\theta = \tfrac{1}{2}\operatorname{cosec}^2\theta - \log\sin\theta,$$

we find

$$u = A'\sin\theta + B'(\cot\theta - \sin\theta\log\tan\tfrac{1}{2}\theta) + \tfrac{1}{6}C_1(\operatorname{cosec}\theta - \sin\theta\log\sin\theta).$$

To make this finite when $\theta = 0$, we must take

$$6B' + C_1 = 0\,;$$

thus we have for the tangential displacement

$$u = A'\sin\theta + \tfrac{1}{6}C_1\left[\tan\tfrac{1}{2}\theta - \sin\theta\log(1+\cos\theta)\right] \dots\dots\dots (30).$$

By (10) we find for the radial displacement

$$w = -A'\cos\theta + \tfrac{1}{6}C_1\left[\cos\theta\log(1+\cos\theta) - 1 + \tfrac{1}{2}\cos\theta\right]$$
$$+ \tfrac{1}{4}\frac{m+n}{m-n}C_1(\cos\theta - \cos\alpha) \quad \dots\dots (31).$$

Hence

$$a\sigma_1 = \tfrac{1}{4}\frac{m+n}{3m-n}C_1(\cos\theta - \cos\alpha) - \tfrac{1}{6}C_1 + \tfrac{1}{12}C_1\cos\theta + \tfrac{1}{6}C_1\frac{1+\sin^2\theta}{1+\cos\theta}$$

$$= \tfrac{1}{4}\frac{m+n}{3m-n}C_1(\cos\theta - \cos\alpha) + \tfrac{1}{12}C_1\frac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}.$$

So $a\sigma_2 = \tfrac{1}{4}\dfrac{m+n}{3m-n}C_1(\cos\theta - \cos\alpha) - \tfrac{1}{12}C_1\dfrac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}.$

Hence $\quad a\left(\sigma_1 + \dfrac{m-n}{2m}\sigma_2\right) = \tfrac{1}{12}\dfrac{m+n}{2m}C_1\dfrac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha} \quad \dots\dots (32)$

when $\theta = \alpha$.

Seeing that ϖ vanishes identically, the second of the boundary conditions (4) is satisfied identically, and the first gives for the edge-traction that must be applied to the bowl

$$A = 2nh\frac{4m}{m+n}\frac{1}{12a}\frac{m+n}{2m}C_1\frac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha}$$

$$= \frac{4nh}{12a}C_1\frac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha},$$

and this is $\qquad A = \tfrac{1}{6}g\rho' a^2\dfrac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha} \quad \dots\dots\dots\dots\dots\dots (33)$

in case the bowl is subject to fluid pressure.

It is easy to verify that the resultant upwards traction is equal to the weight of the liquid, for this resultant is

$$\tfrac{1}{6} g\rho' a^2 \frac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha}(2\pi a\sin\alpha)\sin\alpha,$$

or

$$\tfrac{1}{3} g\rho'\pi a^3 (2+\cos\alpha)(1-\cos\alpha)^2,$$

which is right.

The terms in A' represent a rigid-body displacement. The rim to which the supporting force is applied may be supposed to suffer no tangential displacement; in this case u vanishes when $\theta=\alpha$, and we determine A' by the equation

$$A'\sin\alpha = \frac{g\rho' a^3}{12nh}\left[\sin\alpha\log(1+\cos\alpha)-\tan\tfrac{1}{2}\alpha\right];$$

and the displacements at any point are

$$u = \frac{g\rho' a^3}{12nh}\left\{\left[\log(1+\cos\alpha)-\tfrac{1}{2}\sec^2\tfrac{1}{2}\alpha\right]\sin\theta - \left[\sin\theta\log(1+\cos\theta)-\tan\tfrac{1}{2}\theta\right]\right\} \quad\ldots\ldots(34)$$

along the meridian, and

$$w = \frac{g\rho' a^3}{12nh}\left\{-\left[\log(1+\cos\alpha)-\tfrac{1}{2}\sec^2\tfrac{1}{2}\alpha\right]\cos\theta + \left[\cos\theta\log(1+\cos\theta)-1+\tfrac{1}{2}\cos\theta\right]\right\} - \frac{m+n}{m-n}\frac{g\rho' a^3}{8nh}(\cos\theta-\cos\alpha) \quad\ldots\ldots\ldots\ldots(35)$$

along the normal.

In the particular case of a hemisphere supported by a uniformly stretched vertical membrane in the form of a cylinder, we find that the vertical displacement of the lowest point is

$$g\rho' a^3\left[\tfrac{1}{3}\log 2-\tfrac{1}{2}(m+n)/(m-n)\right]/4nh,$$

and the tension of the membrane is

$$\tfrac{1}{3} g\rho' a^2.$$

Example II.—A bowl supported in the same way as before is deflected by its own weight.

In this problem

$$X=-2g\rho h\sin\theta,\quad Y=0,\quad Z=2g\rho h\cos\theta,$$

$$X'+\tfrac{1}{2}\frac{\partial Z'}{\partial\theta} = -3g\rho h\, a^2\sin\theta/2nh = -\tfrac{3}{2}g\rho a^2\sin\theta/n.$$

Writing this $-\frac{1}{4}C_1 \sin\theta$, for shortness, we have, just as before,

$$\left.\begin{aligned} u &= \quad A'\sin\theta + \tfrac{1}{8}C_1\left[\tan\tfrac{1}{2}\theta - \log(1+\cos\theta)\right] \\ w &= -A'\cos\theta + \tfrac{1}{8}C_1\left[\cos\theta\log(1+\cos\theta) - \tfrac{1}{2} + \cos\theta\right] \\ &\qquad + \tfrac{1}{4}\frac{m+n}{3m-n}\frac{g\rho a^2}{n}\cos\theta \end{aligned}\right\} \quad \ldots\ldots (36)$$

where the last term comes from the Z' term in equation (10).

Hence $$a\sigma_1 = \tfrac{1}{4}\frac{m+n}{3m-n}\frac{g\rho a^2}{n}\cos\theta + \frac{C_1}{12}\frac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}$$

$$a\sigma_2 = \tfrac{1}{4}\frac{m+n}{3m-n}\frac{g\rho a^2}{n}\cos\theta - \frac{C_1}{12}\frac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}.$$

Thus

$$a\left(\sigma_1 + \frac{m-n}{2m}\sigma_2\right) = \tfrac{1}{4}\frac{m+n}{2m}\frac{g\rho a^2}{n}\cos\theta + \frac{C_1}{12}\frac{m+n}{2m}\frac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}.$$

Now the boundary condition gives for the edge-traction

$$A = 4nh\frac{2m}{m+n}\left(\sigma_1 + \frac{m-n}{2m}\sigma_2\right)$$

$$= g\rho ah\cos a + g\rho ah\frac{(2+\cos a)(1-\cos a)}{1+\cos a} \quad \text{when } \theta = a,$$

or $$A = 2g\rho ah/(1+\cos a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (37).$$

The resultant upwards traction is

$$\frac{2g\rho ah}{1+\cos a}2\pi a\sin^2 a$$

$$= 2h\,.\,2\pi a^2(1-\cos a)\,g\rho$$

$$= \text{weight of shell, as it should be.}$$

Example III.—If we change the sign of g, the above analysis applies to the case of a hemispherical bowl, resting with its vertex upwards on a smooth horizontal plane.

We have to put $u = 0$ when $\theta = \frac{1}{2}\pi$; thus

$$A' = ga^2\rho/2n,$$

and the displacements are

$$\left.\begin{aligned} w &= -\frac{ga^2\rho}{2n}\cos\theta - \frac{ga^2\rho}{2n}\left[\cos\theta\log(1+\cos\theta) - 1 + \tfrac{1}{2}\cos\theta\right] \\ &\qquad\qquad - \frac{ga^2\rho}{4n}\,\frac{m+n}{3m-n}\cos\theta \\ u &= \frac{ga^2\rho}{2n}\sin\theta - \frac{ga^2\rho}{2n}\left[\tan\tfrac{1}{2}\theta - \sin\theta\log(1+\cos\theta)\right] \end{aligned}\right\} \quad \ldots(38).$$

Hence, if $2h$ be the thickness, the deflection at the vertex is

$$\left[\tfrac{1}{2} + \log 2 + \tfrac{1}{2}(m+n)/(3m-n)\right] W/8\pi nh \quad \ldots\ldots\ldots\ldots\ldots(39),$$

where W is the weight of the bowl.

A Method of Transformation with the aid of Congruences of a Particular Type. *By* J. Brill, M.A.

[*Read Dec.* 13*th*, 1888.]

1. Suppose that we have a family (A) of surfaces. The orthogonal trajectories of this family will form a congruence (a) of curves. On one of the surfaces belonging to the family (A) draw a family of lines. The curves of the congruence (a) that meet each of these lines will form a surface; and the curves of the congruence (a) that meet all of these lines will form a family (B) of surfaces, which is such that the members of it intersect orthogonally the members of the family (A). The curves of intersection of the members of the family (A) with those of the family (B) will form a congruence (c). This congruence will possess the property that it is possible to draw within it* two families of surfaces, viz. the families (A) and (B), such that the members of the one intersect the members of the other orthogonally. Further, since the family of lines drawn on the selected surface of the family (A) are altogether arbitrary, it is evident that they may be chosen so that at least one other selected property may belong to the congruence. It is, however, conceivable that cases may

* By this expression it is intended that each surface is the locus of some singly [illegible] of the curves of the congruence in question.

arise in which one other property may be selected, the choice of which property may not be altogether arbitrary.

2. We will take α and β, the parameters of the families (A) and (B), as the coordinates of the curves of the congruence (c). Then we have the three equations

$$\left.\begin{aligned} \left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\alpha}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial z}\right)^2 &= k_1^2 \\ \left(\frac{\partial\beta}{\partial x}\right)^2+\left(\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\beta}{\partial z}\right)^2 &= k_2^2 \\ \frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x}+\frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y}+\frac{\partial\alpha}{\partial z}\frac{\partial\beta}{\partial z} &= 0 \end{aligned}\right\} \quad \text{....................(I.).}$$

We proceed to discover what conditions must be satisfied by the families (A) and (B) in order that, if any family (C) of surfaces be drawn within the congruence, it may be always possible to draw another family (D) within the congruence, such that the members of it intersect orthogonally those of the family (C).

Suppose that ξ and η are the parameters of the families (C) and (D). We have the equations

$$\left.\begin{aligned} \left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\xi}{\partial z}\right)^2 &= h_1^2 \\ \left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2+\left(\frac{\partial\eta}{\partial z}\right)^2 &= h_2^2 \\ \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial z}\frac{\partial\eta}{\partial z} &= 0 \end{aligned}\right\} \quad \text{..................(II.).}$$

If we substitute for $\partial\xi/\partial x$, &c., their values in terms of $\partial\xi/\partial\alpha$ and $\partial\xi/\partial\beta$, and make use of equations (I.), we easily deduce

$$\left.\begin{aligned} k_1^2\left(\frac{\partial\xi}{\partial\alpha}\right)^2+k_2^2\left(\frac{\partial\xi}{\partial\beta}\right)^2 &= h_1^2 \\ k_1^2\left(\frac{\partial\eta}{\partial\alpha}\right)^2+k_2^2\left(\frac{\partial\eta}{\partial\beta}\right)^2 &= h_2^2 \\ k_1^2\frac{\partial\xi}{\partial\alpha}\frac{\partial\eta}{\partial\alpha}+k_2^2\frac{\partial\xi}{\partial\beta}\frac{\partial\eta}{\partial\beta} &= 0 \end{aligned}\right\} \quad \text{......................(III.).}$$

If we are given that $\xi = f(\alpha, \beta)$, and if we substitute the values of

$\partial\xi/\partial\alpha$ and $\partial\xi/\partial\beta$ given by this in the third of equations (III.), we see that this equation will enable us to determine η as a function of α and β, provided that k_2/k_1 is expressible as a function of α and β.

The geometrical interpretation of this condition is easy. Draw two consecutive surfaces belonging to each of the families (A) and (B). These will enclose a space or filament whose normal sections at all points of its length are rectangles having their sides in a constant ratio. It follows, when this condition is satisfied, that any two surfaces drawn within the congruence cut at a constant angle all along their line of intersection.

The above reasoning, of course, breaks down for points on the focal surface of the congruence.

3. The simplest case of a congruence belonging to the type discussed in the preceding article, is that of one consisting of parallel straight lines.

A series of cases may be produced in the following manner. In a given plane draw a family of parallel curves, and with these curves for bases draw a family of cylinders having their generators at right angles to the given plane. The intersections of these cylinders with a family of planes parallel to the given one, will constitute a congruence of the type we are considering. The simplest case of this series is that of the parallel straight lines mentioned above. The next in order of simplicity is that of a congruence of circles having their centres disposed along an axis and their planes at right angles to that axis.

A second series of cases may be obtained by drawing a family of parallel curves on a sphere, and through them drawing a family of cones having the centre of the sphere for vertex. If we cut these cones by a family of spheres concentric with the given one, the intersections will constitute a congruence of the required type.

A third series of cases may be obtained by drawing a family of plane curves possessing the property that, if we draw a normal at any point of one of them, the length of the portion of this normal intercepted between the curve and its consecutive is proportional to the distance of the point from a fixed straight line. If we revolve this figure about the fixed straight line, the congruence formed by the curves in their consecutive positions will be of the type in question.

We are not concerned with the question as to whether it is always possible to obtain a congruence of the type in question containing a given family of surfaces, nor with the problem of discovering the

said congruences in cases where it is possible. It is sufficient for our present purpose to know that examples of this type of congruence do exist.

4. Since k_2/k_1 is expressible as a function of a and β, it is evident from equations (III.) that h_2/h_1 may be also so expressed. From the same equations we easily deduce

$$\frac{k_1\dfrac{\partial\xi}{\partial a}}{k_2\dfrac{\partial\eta}{\partial\beta}} = -\frac{k_2\dfrac{\partial\xi}{\partial\beta}}{k_1\dfrac{\partial\eta}{\partial a}} = \pm\frac{\left\{k_1^2\left(\dfrac{\partial\xi}{\partial a}\right)^2+k_2^2\left(\dfrac{\partial\xi}{\partial\beta}\right)^2\right\}^{\frac{1}{2}}}{\left\{k_1^2\left(\dfrac{\partial\eta}{\partial a}\right)^2+k_2^2\left(\dfrac{\partial\eta}{\partial\beta}\right)^2\right\}^{\frac{1}{2}}} = \pm\frac{h_1}{h_2}.$$

Thus we should have either

$$h_2k_1\frac{\partial\xi}{\partial a} = h_1k_2\frac{\partial\eta}{\partial\beta} \quad\text{and}\quad h_2k_2\frac{\partial\xi}{\partial\beta} = -h_1k_1\frac{\partial\eta}{\partial a},$$

or

$$h_2k_1\frac{\partial\xi}{\partial a} = -h_1k_2\frac{\partial\eta}{\partial\beta} \quad\text{and}\quad h_2k_2\frac{\partial\xi}{\partial\beta} = h_1k_1\frac{\partial\eta}{\partial a}.$$

These two forms are virtually identical. In developing our theory we will make use of the first form. However, in discussing any particular case, it would be necessary to ascertain carefully which parameter should be taken for ξ and which for η.

Now h_2/h_1 and k_2/k_1 are both functions of a and β. We will express them by the symbols u and v respectively. Then our equations take the form

$$u\frac{\partial\xi}{\partial a} = v\frac{\partial\eta}{\partial\beta} \quad\text{and}\quad uv\frac{\partial\xi}{\partial\beta} = -\frac{\partial\eta}{\partial a}.$$

Now

$$\begin{aligned} ud\xi+id\eta &= u\left\{\frac{\partial\xi}{\partial a}da+\frac{\partial\xi}{\partial\beta}d\beta\right\}+i\left\{\frac{\partial\eta}{\partial a}da+\frac{\partial\eta}{\partial\beta}d\beta\right\} \\ &= u\frac{\partial\xi}{\partial a}\left(da+i\frac{d\beta}{v}\right)+i\frac{\partial\eta}{\partial a}\left(da+i\frac{d\beta}{v}\right) \\ &= \left\{u\frac{\partial\xi}{\partial a}+i\frac{\partial\eta}{\partial a}\right\}\left(da+i\frac{d\beta}{v}\right). \end{aligned}$$

Therefore

$$\frac{ud\xi+id\eta}{vda+id\beta} = \frac{1}{v}\left\{u\frac{\partial\xi}{\partial a}+i\frac{\partial\eta}{\partial a}\right\} = \frac{\partial\eta}{\partial\beta}-iu\frac{\partial\xi}{\partial\beta}.$$

Thus the value of the expression

$$\frac{u\,d\xi+i\,d\eta}{v\,da+i\,d\beta}$$

is independent of the value of the ratio $da : d\beta$. The same will be true of the expression

$$\frac{(m+in)(u\,d\xi+i\,d\eta)}{(p+iq)(v\,da+i\,d\beta)}.$$

Writing $f=m+in$ and $g=p+iq$, we will seek to determine f and g so that the expressions $f(u\,d\xi+i\,d\eta)$ and $g(v\,da+i\,d\beta)$ may be perfect differentials. The necessary conditions for this will be

$$\frac{\partial}{\partial\eta}(fu)=i\frac{\partial f}{\partial\xi} \quad\text{and}\quad \frac{\partial}{\partial\beta}(gv)=i\frac{\partial g}{\partial a}.$$

These equations will enable us to obtain suitable forms for f and g, and the solution of each will involve an arbitrary function. If we now write

$$dw=f(u\,d\xi+i\,d\eta) \quad\text{and}\quad d\zeta=g(v\,da+i\,d\beta),$$

$dw/d\zeta$ will possess a single definite value; and, if we further write $w=\lambda+i\mu$ and $\zeta=\gamma+i\delta$, we see that the value of the expression

$$\frac{d\lambda+i\,d\mu}{d\gamma+i\,d\delta}$$

is independent of the value of the ratio $d\gamma : d\delta$. This necessitates the relations

$$\frac{\partial\lambda}{\partial\gamma}=\frac{\partial\mu}{\partial\delta} \quad\text{and}\quad \frac{\partial\lambda}{\partial\delta}=-\frac{\partial\mu}{\partial\gamma}.$$

But these are the conditions that $\lambda+i\mu$ may be expressible as a function of $\gamma+i\delta$. Thus we see that we can draw within the congruence an indefinite number of systems containing two orthogonal families of surfaces, which possess properties somewhat analogous to those possessed by plane systems consisting of two orthogonal families of equipotential curves. All the systems of this character may be obtained from the expression

$$\int(p+iq)(v\,da+i\,d\beta)$$

with the aid of the general integral of the equation

$$\frac{\partial}{\partial\beta}(gv) = i\frac{\partial g}{\partial a}.*$$

It is to be noted, however, that the families of surfaces of which these systems consist are not necessarily equipotential.

5. We are now in a position, with the aid of congruences of the type we have been discussing, to develop a method of transformation in space of three dimensions analogous to the method of transformation by means of conjugate functions in a plane. In this method the curves of the congruence take the place of the points in the plane, and the surfaces of the congruence take the place of curves in the plane.

If we take γ and δ as the parameters of the two families of surfaces constituting a system of the type discussed at the end of the preceding article, and if we take λ and μ as the parameters of the two families constituting another such system, we have $\lambda+i\mu = F(\gamma+i\delta)$. Now take a congruence exactly like the one in which these systems are drawn; and make those families of surfaces within it, the ones exactly like which in the first congruence belong to the parameters λ and μ, correspond to the parameters γ and δ. Then any surface drawn within the first congruence will be transformed into a different surface within the second congruence.

It is easy to prove that any pair of transformed surfaces will cut at the same angle as the original pair of surfaces. Thus, from the equation $\qquad (p+iq)(vda+id\beta) = d\gamma+id\delta,$

we deduce $\qquad pvda-qd\beta = d\gamma$ and $qvda+pd\beta = d\delta$;

from which it follows that

$$\frac{\partial\gamma}{\partial a} = pv,\quad \frac{\partial\gamma}{\partial\beta} = -q,\quad \frac{\partial\delta}{\partial a} = qv,\quad \frac{\partial\delta}{\partial\beta} = p.$$

Therefore

$$k_1^2\left(\frac{\partial\gamma}{\partial a}\right)^2+k_2^2\left(\frac{\partial\gamma}{\partial\beta}\right)^2 = k_1^2\left(\frac{\partial\delta}{\partial a}\right)^2+k_2^2\left(\frac{\partial\delta}{\partial\beta}\right)^2 = (p^2+q^2)\,k_2^2 = k^2 \text{ say.}$$

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* The referee points out that all the congruences of the type considered can be obtained from the equations

$$\left(\frac{\partial\gamma}{\partial x}\right)^2+\left(\frac{\partial\gamma}{\partial y}\right)^2+\left(\frac{\partial\gamma}{\partial z}\right)^2 = \left(\frac{\partial\delta}{\partial x}\right)^2+\left(\frac{\partial\delta}{\partial y}\right)^2+\left(\frac{\partial\delta}{\partial z}\right)^2$$

and $$\frac{\partial\gamma}{\partial x}\frac{\partial\delta}{\partial x}+\frac{\partial\gamma}{\partial y}\frac{\partial\delta}{\partial y}+\frac{\partial\gamma}{\partial z}\frac{\partial\delta}{\partial z} = 0.$$

Thus, if ρ and σ be the parameters of the two surfaces in question,

then
$$\left(\frac{\partial\rho}{\partial x}\right)^2+\left(\frac{\partial\rho}{\partial y}\right)^2+\left(\frac{\partial\rho}{\partial z}\right)^2=k^2\left\{\left(\frac{\partial\rho}{\partial\gamma}\right)^2+\left(\frac{\partial\rho}{\partial\delta}\right)^2\right\},$$

$$\left(\frac{\partial\sigma}{\partial x}\right)^2+\left(\frac{\partial\sigma}{\partial y}\right)^2+\left(\frac{\partial\sigma}{\partial z}\right)^2=k^2\left\{\left(\frac{\partial\sigma}{\partial\gamma}\right)^2+\left(\frac{\partial\sigma}{\partial\delta}\right)^2\right\},$$

$$\frac{\partial\rho}{\partial x}\frac{\partial\sigma}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial\sigma}{\partial y}+\frac{\partial\rho}{\partial z}\frac{\partial\sigma}{\partial z}=k^2\left\{\frac{\partial\rho}{\partial\gamma}\frac{\partial\sigma}{\partial\gamma}+\frac{\partial\rho}{\partial\delta}\frac{\partial\sigma}{\partial\delta}\right\}.$$

Hence, if θ be the angle at which the surfaces cut

$$\cos\theta=\frac{\dfrac{\partial\rho}{\partial\gamma}\dfrac{\partial\sigma}{\partial\gamma}+\dfrac{\partial\rho}{\partial\delta}\dfrac{\partial\sigma}{\partial\delta}}{\left\{\left(\dfrac{\partial\rho}{\partial\gamma}\right)^2+\left(\dfrac{\partial\rho}{\partial\delta}\right)^2\right\}^{\frac{1}{2}}\left\{\left(\dfrac{\partial\sigma}{\partial\gamma}\right)^2+\left(\dfrac{\partial\sigma}{\partial\delta}\right)^2\right\}^{\frac{1}{2}}}.$$

But we have

$$\left(\frac{\partial\lambda}{\partial\gamma}\right)^2+\left(\frac{\partial\lambda}{\partial\delta}\right)^2=\left(\frac{\partial\mu}{\partial\gamma}\right)^2+\left(\frac{\partial\mu}{\partial\delta}\right)^2=s^2 \text{ say,}$$

and
$$\frac{\partial\lambda}{\partial\gamma}\frac{\partial\mu}{\partial\gamma}+\frac{\partial\lambda}{\partial\delta}\frac{\partial\mu}{\partial\delta}=0.$$

Therefore
$$\left(\frac{\partial\rho}{\partial\gamma}\right)^2+\left(\frac{\partial\rho}{\partial\delta}\right)^2=s^2\left\{\left(\frac{\partial\rho}{\partial\lambda}\right)^2+\left(\frac{\partial\rho}{\partial\mu}\right)^2\right\},$$

$$\left(\frac{\partial\sigma}{\partial\gamma}\right)^2+\left(\frac{\partial\sigma}{\partial\delta}\right)^2=s^2\left\{\left(\frac{\partial\sigma}{\partial\lambda}\right)^2+\left(\frac{\partial\sigma}{\partial\mu}\right)^2\right\},$$

and
$$\frac{\partial\rho}{\partial\gamma}\frac{\partial\sigma}{\partial\gamma}+\frac{\partial\rho}{\partial\delta}\frac{\partial\sigma}{\partial\delta}=s^2\left\{\frac{\partial\rho}{\partial\lambda}\frac{\partial\sigma}{\partial\lambda}+\frac{\partial\rho}{\partial\mu}\frac{\partial\sigma}{\partial\mu}\right\}.$$

Therefore

$$\cos\theta=\frac{\dfrac{\partial\rho}{\partial\lambda}\dfrac{\partial\sigma}{\partial\lambda}+\dfrac{\partial\rho}{\partial\mu}\dfrac{\partial\sigma}{\partial\mu}}{\left\{\left(\dfrac{\partial\rho}{\partial\lambda}\right)^2+\left(\dfrac{\partial\rho}{\partial\mu}\right)^2\right\}^{\frac{1}{2}}\left\{\left(\dfrac{\partial\sigma}{\partial\lambda}\right)^2+\left(\dfrac{\partial\sigma}{\partial\mu}\right)^2\right\}^{\frac{1}{2}}};$$

and the second side of this equation denotes the cosine of the angle at which the transformed surfaces cut.

Further, it is easily seen that we may utilise our results to obtain a correspondence between two different congruences of the type in question, which is exactly similar in character to the geographical correspondence of two surfaces.

6. If we apply our method to the simplest congruence of the specified type, viz., the one consisting of parallel straight lines, it reduces to the ordinary method of transformation by means of functions of a complex variable as applied to the plane. Also, if we apply our method to the congruence of circles having their centres disposed along an axis, and their planes at right angles to this axis, we shall have a method applicable to surfaces of revolution having a common axis.

In conclusion, I may state that I think I possess a clue to the establishment of a correspondence between any two congruences whatsoever, and that I hope to make a communication to the Society on this subject at some future time.

Thursday, January 10th, 1889.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Mr. G. H. Bryan, M.A., St. Peter's College, Cambridge, and Mr. W. W. Taylor, M.A., late Scholar of Queen's College, Oxford, were elected members, and Miss Meyer was admitted into the Society.*

The auditor (Mr. Heppel) made his report, which on the motion of Sir J. Cockle, seconded by Rev. T. C. Simmons, was adopted. Upon the motion of Mr. Basset, seconded by Dr. Glaisher, the Treasurer's report was adopted.

Subsequently a vote of thanks was passed to the Auditor on the motion of Major Macmahon, R.A., seconded by Dr. Glaisher.

Mr. Basset made a few remarks on the Steady Motion and Stability of Dynamical Systems.

Dr. Glaisher gave several forms of expression of Bernoulli's Numbers derived from the consideration of Lemniscate properties.

The President read a paper on "Results of Ternary Quadric Operators on Products of Forms of any Orders" (Sir J. Cockle in the Chair).

Mr. Jenkins read a note by Mr. Christie "On a Theorem in Combinations."

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* By an oversight it was omitted to be stated that Mr. R. W. Hogg was admitted into the Society at the November meeting.

Presents received during the Recess :—

"Proceedings of the Royal Society," No. 272.

"The Educational Times" for January.

"Proceedings of the Cambridge Philosophical Society," Vol. VI., Part IV.; Cambridge, 1888.

"Mathematics from the 'Educational Times,'" Vol. XLIX.

"Royal Irish Academy—Proceedings," Vol. I., No. 1 (Third Series), "Transactions," Vol. XXIX., Parts III. and IV.

"Bulletin de la Société Mathématique de France," Tome XVI., No. 5; Paris, 1888.

"Leçons sur la Théorie Générale des Surfaces," Deuxième Partie, par Gaston Darboux, 8vo; Paris, 1889.

"Beiblätter zu den Annalen der Physik und Chemie," Band XII., Stück 11; Leipzig, 1888.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VIII., No. 6; Coimbra, 1887.

"Rendiconti del Circolo Matematico di Palermo," Tomo II., Fasc. VI., Nov.-Dic., 1888.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Nos. 71 and 72; Firenze.

"Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti," Tomo V., Ser. VI., Disp. 10; Tomo VI., Ser. VI., Disp. 1—9.

"Journal of the College of Science," Imperial University, Japan, Vol. II., Part IV.; Tokio, Japan, 1888.

"American Journal of Mathematics," Vol. XI., No. 2.

Results of Ternary Quadric Operators on Products of Forms of any Orders. *By* J. J. Walker, F.R.S.

[*Read Jan.* 10*th*, 1889.]

I. *Abstract.*

Considering any ternary form expressed in the quadric shape

$$(\text{abcfgh} \between xyz)^2 \equiv s,$$

and the explicit xyz replaced by symbols of differentiation

$$x \text{ by } \partial_x, \quad y \text{ by } \partial_y, \quad z \text{ by } \partial_z,$$

the operator so obtained may be termed a "pure" quadric operator; but, if the variables are replaced

$$x \text{ by } n\partial_y - m\partial_z, \quad y \text{ by } l\partial_z - n\partial_x, \quad z \text{ by } m\partial_x - l\partial_y,$$

l, m, n being the constant coefficients of a linear form, say,

$$L \equiv lx + my + nz,$$

the latter may be termed a "mixed" quadric operator.

The objects of the present Note are to show:—

(1) How the results of the "pure" operator on the product of any two forms—say, v, w—of orders q, r respectively, may be exhibited in three distinct shapes, viz.—

(α) The "crude" shape;

(β) That from which the first differential coefficients of v, w have been eliminated by certain substitutions giving rise to terms reinforcing those of the "crude" shape, in which v, w are either multiplied by the result of operating on the other, the residue being expressed in the quadric shape

$$(\mathrm{b}C_1 + \mathrm{c}B_1 - 2\mathrm{f}F_1, \ldots\ \mathrm{g}H_1 + \mathrm{h}G_1 - \mathrm{a}F_1 - \mathrm{f}A_1, \ldots \between xyz)^2 \ldots\ldots \text{(i.)},$$

wherein $A_1 \ldots F_1 \ldots$ are the coefficients of the contravariant quadric of v, w, themselves expressed in the quadric form;

(γ) And a third shape from which either v or w—or both, if forms of the same order—have been eliminated:

(2) How the results (β), (γ) may be extended to cases where the operand is the product of more than two forms:

(3) How the corresponding results may be best expressed when the "mixed" operator is employed on the product of two forms, viz., the result (β) becomes the sum of five terms in three of which s, v, w are each multiplied by the quadric contravariant of the other two; a fourth term in which L^2 is multiplied by the invariant of svw,

$$\mathrm{a}A_1 + \ldots + 2\mathrm{f}F_1 + \ldots;$$

and lastly, a term in which L is multiplied by the sum of the products of the first differential coefficients of s with respect to x, y, z by those of the quadric contravariant of v, w with respect to l, m, n, respectively:

(4) How the last result is extended to the product of more than two forms.

The only lengthy piece of work involved being the development of the expression (i.) above, when the product of two functions is substituted for either v or w in it, the investigation of this is made a preliminary Lemma, which has a sort of intrinsic interest of its own.

II. *Preliminary.*

Supposing, then, v, w to be ternary forms of orders p, q, and representing, for brevity,

$$\left.\begin{array}{l}\partial v/p\,\partial x \text{ by } v_1, \quad \partial w/q\,\partial x \text{ by } w_1, \quad \partial v/p\,\partial y \text{ by } v_2 \ldots \\ \partial^2 v/p\,.\,p-1\,\partial x^2 \text{ by } a', \ldots\ \partial^2 v/p\,.\,p-1\,\partial y\,\partial z \text{ by } f' \ldots \\ \partial^2 w/q\,.\,q-1\,\partial x^2 \text{ by } a'', \ldots\ \partial^2 w/q\,.\,q-1\,\partial y\,\partial z \text{ by } f'' \ldots\end{array}\right\} \ldots\ldots\ldots\ldots(1),$$

$$\begin{aligned}2v_1 w_1 &= 2\,(a'x+h'y+g'z)(a''x+h''y+g''z)\\ &= a''v+a'w\\ &\quad -(c'a''+c''a'-2g'g'')\,z^2-(a'b''+a''b'-2h'h'')\,y^2\\ &\quad +2\,(g'h''+g''h'-a'f''-a''f')\,yz\end{aligned}$$

or, say, $$= a''v+a'w-B_1 z^2-C_1 y^2+2F_1 yz \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and, similarly,

$$v_2 w_3+v_3 w_2 = f''v+f'w+A_1 yz+F_1 x^2-G_1 xy-H_1 zx\ldots\ldots\ldots(3),$$

where $A_1 = b'c''+b''c'-2f'f''$, ... $H_1 = f'g''+f''g'-c'h''-c''h'$;

with analogous values for $2v_2 w_2 \ldots v_3 w_1+v_1 w_3 \ldots$.

Next, let u, v, w be three ternary forms (in xyz) of orders p, q, r, respectively, and

$$A_1 = \left(\frac{\partial^2 v}{\partial y^2}\,\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 v}{\partial z^2}\,\frac{\partial^2 w}{\partial y^2}-2\,\frac{\partial^2 v}{\partial y\,\partial z}\,\frac{\partial^2 w}{\partial y\,\partial z}\right)\Big/ qr\,(q-1)(r-1),$$

$$B_1 = \ldots, \qquad C_1 = \ldots,$$

$$F_1 = \left(\frac{\partial^2 v}{\partial z\partial x}\,\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x\partial y}\,\frac{\partial^2 w}{\partial z\,\partial x}-\frac{\partial^2 v}{\partial x^2}\,\frac{\partial^2 w}{\partial y\,\partial z}-\frac{\partial^2 v}{\partial y\,\partial z}\,\frac{\partial^2 w}{\partial x^2}\right)$$

$$\Big/ qr\,(q-1)(r-1),$$

$$G_1 = \ldots, \qquad H_1 = \ldots,$$

while $A_2 \ldots H_2$ stand for the analogous functions of w, u; and $A_3 \ldots H_3$ for those of u, v.

Further, let any other form s be written in the shape

$$s \equiv \mathrm{a}x^2+\mathrm{b}y^2+\mathrm{c}z^2+2\mathrm{f}yz+2\mathrm{g}zx+2\mathrm{h}xy\,;$$

then, if in the formula

$$[s, v, w] \equiv (\mathrm{b}C_1+\mathrm{c}B_1-2\mathrm{f}F_1)\,x^2+\ldots+2\,(\mathrm{g}H_1+\mathrm{h}G_1-\mathrm{a}F_1-\mathrm{f}A_1)\,yz+\ldots$$

the product uv be substituted for v, it may be shown that

$$-\left[s, uv, w\right] = pqw\left[s, u, v\right]/(p+q)(p+q-1)$$
$$-pv\left[s, w, u\right]/(p+q)-qu\left[s, v, w\right]/(p+q) \dots\dots\dots (4).$$

To prove this:

$$\frac{\partial^2(uv)}{\partial x^2} = u\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^2 u}{\partial x^2}+2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$$

$$(2), \dots\dots\dots\dots = (p+q-1)(qua'+pva)-pq\left(B_3z^2+C_3y^2-2F_3yz\right);$$

and (3),

$$\frac{\partial^2(uv)}{\partial y\partial z} = (p+q-1)(quf'+pvf)-pq\left(-A_3yz-F_3x^2+G_3xy+H_3xy\right),$$

with analogous values for $\partial^2(uv)/\partial y^2 \dots \partial^2(uv)/\partial x\partial y$.

If now $A \dots F \dots$ (unmarked) are what $A_1 \dots F_1$ respectively become by the substitution of uv for v, and, consequently, of $p+q$ for q, writing, as before,

$w_1 \dots$ for $\partial w/r\,\partial x \dots$, $a'' \dots f'' \dots$ for $\partial^2 w/r.r-1\,\partial x^2 \dots$, $\partial^2 w/r.r-1\,\partial y\partial z \dots$,

$$(p+q)(p+q-1)\,A = (p+q-1)(quA_1+pvA_2)-pqwA_3$$
$$-pq\left(A_3a''+\dots+2F_3f''+\dots\right)x^2$$
$$-2pq\left(A_3w_1+H_3w_2+G_3w_3\right)x,$$

$$(p+q)(p+q-1)\,B = (p+q-1)(quB_1+pvB_2)-pqwB_3$$
$$-pq\left(A_3a''+\dots+2F_3f''+\dots\right)y^2$$
$$-2pq\left(H_3w_1+B_3w_2+F_3w_3\right)y,$$

$$(p+q)(p+q-1)\,C = (p+q-1)(quC_1+pvC_2)-pqwC_3$$
$$-pq\left(A_3a''+\dots+2F_3f''+\dots\right)z^2$$
$$-2pq\left(G_3w_1+F_3w_2+C_3w_3\right)z,$$

$$(p+q)(p+q-1)\,F = (p+q-1)(quF_1+pvF_2)-pqwF_3$$
$$-pq\left(A_3a''+\dots+2F_3f''+\dots\right)yz$$
$$-pq\left\{\left(G_3w_1+F_3w_2+C_3w_3\right)y+\left(H_3w_1+B_3w_2+F_3w_3\right)z\right\},$$

$$(p+q)(p+q-1)\,G = (p+q-1)(quG_1+pvG_2)-pqwG_3$$
$$-pq\left(A_3a''+\dots+2F_3f''+\dots\right)zx$$
$$-pq\left\{\left(A_3w_1+H_3w_2+G_3w_3\right)z+\left(G_3w_1+F_3w_2+C_3w_3\right)x\right\},$$

$$(p+q)(p+q-1)\,H = (p+q-1)(quH_1+pvH_2)-pqwH_3$$
$$-pq\,(A_3a''+\ldots+2F_3f''+\ldots)\,xy$$
$$-pq\,\{(H_3w_1+B_3w_2+F_3w_3)\,x+(A_3w_1+H_3w_2+G_3w_3)\,y\}.$$

From the above equalities

$$-(p+q)(p+q-1)(B\mathrm{c}+C\mathrm{b}-2F\mathrm{f})$$
$$=-(p+q-1)\{qu\,(\mathrm{b}C_1+\mathrm{c}B_1-2\mathrm{f}F_1)+pv\,(\mathrm{b}C_2+\mathrm{c}B_2-2\mathrm{f}F_2)\}$$
$$+pqw\,(\mathrm{b}C_3+\mathrm{c}B_3-2\mathrm{f}F_3)$$
$$+pq\,(A_3a''+\ldots+2F_3f''+\ldots)(\mathrm{b}z^2+\mathrm{c}y^2-2\mathrm{f}yz)$$
$$+2pq\,(H_3w_1+B_3w_2+F_3w_3)(\mathrm{f}z-\mathrm{c}y)+2pq\,(G_3w_1+F_3w_2+C_3w_3)(\mathrm{f}y-\mathrm{b}z),$$

$$-(p+q)(p+q-1)(\mathrm{f}G+\mathrm{g}F-\mathrm{c}H-\mathrm{h}C)$$
$$=-(p+q-1)\{qu(\mathrm{f}G_1+\mathrm{g}F_1-\mathrm{c}H_1-\mathrm{h}C_1)+pv(\mathrm{f}G_2+\mathrm{g}F_2-\mathrm{c}H_2-\mathrm{h}C_2)\}$$
$$+pqw\,(\mathrm{f}G_3+\mathrm{g}F_3-\mathrm{c}H_3-\mathrm{h}C_3)$$
$$+pq\,(A_3a''+\ldots+2F_3f''+\ldots)(-\mathrm{c}xy+\mathrm{f}zx+\mathrm{g}yz-\mathrm{h}z^2)$$
$$+pq\,(A_3w_1+H_3w_2+G_3w_3)(\mathrm{c}y-\mathrm{f}z)+pq\,(H_3w_1+B_3w_2+F_3w_3)(\mathrm{c}x-\mathrm{g}z)$$
$$+pq\,(G_3w_1+F_3w_2+Cw_3)(-\mathrm{f}x-\mathrm{g}y+2\mathrm{h}z),$$

$$-(p+q)(p+q-1)(\mathrm{h}F+\mathrm{f}H-\mathrm{b}G-\mathrm{g}B)$$
$$=-(p+q-1)\{qu(\mathrm{h}F_1+\mathrm{f}H_1-\mathrm{b}G_1-\mathrm{g}B_1)+pv(\mathrm{h}F_2+\mathrm{f}H_2-\mathrm{b}G_2-\mathrm{g}B_2)\}$$
$$+pqw\,(\mathrm{h}F_1+\mathrm{f}H_1-\mathrm{b}G_1-\mathrm{g}B_1)$$
$$+pq\,(A_3a''+\ldots+2F_3f''+\ldots)(-\mathrm{b}zx+\mathrm{f}xy-\mathrm{g}y^2+\mathrm{h}yz)$$
$$+pq\,(A_3w_1+H_3w_2+G_3w_3)(\mathrm{b}z-\mathrm{f}y)+pq\,(H_3w_1+B_3w_2+F_3w_3)$$
$$\times(-\mathrm{f}x+2\mathrm{g}y-\mathrm{h}z)+pq\,(G_3w_1+F_3w_2+C_3w_3)(\mathrm{b}x-\mathrm{h}y).$$

The sum of the last three equalities, multiplied by x, y, z, respectively, is

$$-(p+q)(p+q-1)\{(\mathrm{b}C+\mathrm{c}B-2\mathrm{f}F)\,x+(\mathrm{f}G+\mathrm{g}F-\mathrm{c}H-\mathrm{h}C)\,y$$
$$+(\mathrm{h}F+\mathrm{f}H-\mathrm{b}G-\mathrm{g}B)\,z\}$$
$$=-(p+q-1)\,qu\{(\mathrm{b}C_1+\mathrm{c}B_1-2\mathrm{f}F_1)\,x+(\mathrm{f}G_1+\mathrm{g}F_1-\mathrm{c}H_1-\mathrm{h}C_1)\,y$$
$$+(\mathrm{h}F_1+\mathrm{f}H_1-\mathrm{b}G_1-\mathrm{g}B_1)\,z\}$$
$$-(p+q-1)\,pv\{(\mathrm{b}C_2+\ldots)\,x+(\mathrm{f}G_2+\ldots)\,y+(\mathrm{h}F_2+\ldots)\,z)\}$$
$$+\qquad pqw\{(\mathrm{b}C_3+\ldots)\,x+(\mathrm{f}G_3+\ldots)\,y+(\mathrm{h}F_3+\ldots)\,z\}$$
$$+pq\{(A_3w_1+\ldots)(\mathrm{b}z^2+\mathrm{c}y^2-2\mathrm{f}yz)+(H_3w_1+\ldots)(-\mathrm{c}xy+\mathrm{f}zx+\mathrm{g}yz-\mathrm{h}z^2)$$
$$+(G_3w_1+\ldots)(-\mathrm{b}zx+\mathrm{f}xy-\mathrm{g}y^2+\mathrm{h}yz)\}.$$

Analogously,

$$-(p+q)(p+q-1)\{(\mathrm{f}G+\mathrm{g}F-\mathrm{c}H-\mathrm{h}C)\,x+(\mathrm{c}A+\mathrm{a}C-2\mathrm{g}G)\,y$$
$$+(\mathrm{g}H+\mathrm{h}G-\mathrm{a}F-\mathrm{f}A)\,z\}$$
$$=-(p+q-1)\,qu\{(\mathrm{f}G_1+\ldots)\,x+(\mathrm{c}A_1+\ldots)\,y+(\mathrm{g}H_1+\ldots)\,z\}$$
$$-(p+q-1)\,pv\{(\mathrm{f}G_2+\ldots)\,x+(\mathrm{c}A_2+\ldots)\,y+(\mathrm{g}H_2+\ldots)\,z\}$$
$$+\qquad pqw\{(\mathrm{f}G_3+\ldots)\,x+(\mathrm{c}A_3+\ldots)\,y+(\mathrm{g}H_3+\ldots)\,z\}$$
$$+pq\{(A_3w_1+\ldots)(-\mathrm{c}xy+\mathrm{f}zx+\mathrm{g}yz-\mathrm{h}z^2)$$
$$+(H_3w_1+\ldots)(\mathrm{c}x^2+\mathrm{a}z^2-2\mathrm{g}zx)+(G_3w_1+\ldots)(-\mathrm{a}yz-\mathrm{f}x^2+\mathrm{g}xy+\mathrm{h}zx)\},$$

and

$$-(p+q)(p+q-1)\{\mathrm{h}F+\mathrm{f}H-\mathrm{b}G-\mathrm{g}B)\,x+(\mathrm{g}H+\mathrm{h}G-\mathrm{a}F-\mathrm{f}A)\,y$$
$$+(\mathrm{a}B+\mathrm{b}A-2\mathrm{h}H)\,z\}$$
$$=-(p+q-1)\,qu\{(\mathrm{h}F_1+\ldots)\,x+(\mathrm{g}H_1+\ldots)\,y+(\mathrm{a}B_1+\ldots)\,z\}$$
$$-(p+q-1)\,pv\{(\mathrm{h}F_2+\ldots)\,x+(\mathrm{g}H_2+\ldots)\,y+(\mathrm{a}B_2+\ldots)\,z\}$$
$$+\qquad pqw\{(\mathrm{h}F_3+\ldots)\,x+(\mathrm{g}H_3+\ldots)\,y+(\mathrm{a}B_3+\ldots)\,z\}$$
$$+pq\{(A_3w_1+\ldots)(-\mathrm{b}zx+\mathrm{f}xy-\mathrm{g}y^2+\mathrm{h}yz)$$
$$+(H_3w_1+\ldots)(-\mathrm{a}yz-\mathrm{f}x^2+\mathrm{g}xy+\mathrm{h}zx)+(G_3w_1+\ldots)(\mathrm{a}y^2+\mathrm{b}x^2-2\mathrm{h}xy)\}.$$

Finally, the sum of the last three equalities, multiplied by x, y, z, respectively, is

$$-(p+q)(p+q-1)\{(\mathrm{b}C+\mathrm{c}B-2\mathrm{f}F)\,x^2+\ldots$$
$$\ldots+2\,(\mathrm{g}H+\mathrm{h}G-\mathrm{a}F-\mathrm{f}A)\,yz+\ldots\}$$
$$=-(p+q-1)\,qu\{(\mathrm{b}C_1+\ldots)\,x^2+\ldots+2\,(\mathrm{g}H_1+\ldots)\,yz+\ldots\}$$
$$-(p+q-1)\,pv\{(\mathrm{b}C_2+\ldots)\,x^2+\ldots+2\,(\mathrm{g}H_2+\ldots)\,yz+\ldots\}$$
$$+\qquad pqw\{(\mathrm{b}C_3+\ldots)\,x^2+\ldots+2\,(\mathrm{g}H_3+\ldots)\,yz+\ldots\}\;\ldots.(4\ bis),$$

which, divided by $(p+q)(p+q-1)$, is the result (4), p. 113.

III. *Results of operating on vw, uvw ... with* $(\mathrm{abcfgh}\between\partial_x,\partial_y,\partial_z)^2$.

Still employing the notation (1) p. 112, the "crude" shape of

$$(\mathrm{a}\ldots\mathrm{f}\ldots\between\partial_x,\partial_y,\partial_z)^2\,(vw)$$
$$=r\,(r-1)\,v\,(\mathrm{a}a''+\ldots+2\mathrm{f}f''+\ldots)+q\,(q-1)\,w\,(\mathrm{a}a'+\ldots)$$
$$+2qr\,\{\mathrm{a}v_1w_1+\ldots+\mathrm{f}\,(v_2w_3+v_3w_2)+\ldots\}\;\ldots\ldots\ldots\ldots(5).$$

Substituting in this for $2v_1w_1\ldots v_2w_3+v_3w_2\ldots$ from (2), (3), p. 112,

$$(\mathrm{a}\ldots\mathrm{f}\ldots\between\partial_x\partial_y\partial_z)^2(vw) = (q+r-1)\left\{rv(\mathrm{a}a''+\ldots)+qw(\mathrm{a}a'+\ldots)\right\}$$
$$-qr\left\{\mathrm{b}C_1+\mathrm{c}B_1-2\mathrm{f}F_1)x^2+\ldots+2(\mathrm{g}H_1+\mathrm{h}G_1-\mathrm{a}F_1-\mathrm{f}A_1)yz+\ldots\right\} \quad \ldots\ldots\ldots\ldots(6),$$

which is the shape (β) of the Abstract.

Between (5), (6), either of the terms

$$v(\mathrm{a}a''+\ldots) \quad \text{or} \quad w(\mathrm{a}a'+\ldots)$$

may be eliminated, or both if v, w are of the same order, *i.e.*, if $q=r$; and so the shape (γ) of the Abstract be obtained.

Now, suppose the product uv to be substituted for v in (6), u being a form of order p; then, q becoming $p+q$,

$$(\mathrm{a}\ldots\mathrm{f}\ldots\between\partial_x\partial_y\partial_z)^2(uvw)$$
$$=(p+q+r-1)\left\{ruv(\mathrm{a}a''+\ldots)+w(\mathrm{a}\ldots\mathrm{f}\ldots\between\partial_x\partial_y\partial_z)^2(uv)/(p+q-1)\right\}$$
$$-(p+q)r\left\{(\mathrm{b}C+\mathrm{c}B-2\mathrm{f}F)x^2+\ldots\right\},$$

wherein $A\ldots F\ldots$ are what $A_1\ldots F_1\ldots$ of (6) become when v is replaced by uv.

Hence, by (5), (4), using the notation of p. 112,

$$(\mathrm{a}\ldots\mathrm{f}\ldots\between\partial_x\partial_y\partial_z)^2(uvw)$$
$$=(p+q+r-1)\left[\left\{pvw(\mathrm{a}a+\ldots)+qwu(\mathrm{a}a'+\ldots)+ruv(\mathrm{a}a''+\ldots)\right\}\right.$$
$$\left.-pq[s,u,v]/(p+q-1)\right]$$
$$+pqrw[s,u,v]/(p+q-1)-rqu[s,v,w]-rpv[s,w,u]$$
$$=(p+q+r-1)\left\{pvw(\mathrm{a}a+\ldots)+qwu(\mathrm{a}a'+\ldots)+ruv(\mathrm{a}a''+\ldots)\right\}$$
$$-qru[s,v,w]-rpv[s,w,u]-pqw[s,u,v] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

viz., the coefficient of $[s, u, v]$

$$=\left\{-(p+q+r-1)pq+pqr\right\}/(p+q-1)=-pq.$$

By similar steps, the result of operating on the product of four functions may be derived from the case when the operand is the product of three.

Let t be a form of order k, and let

$$\partial^2 t/k.k-1.\partial x^2 = a''' \ldots \partial^2 t/k.k-1.\partial y\partial z = f''' \ldots,$$

then substituting tu for u, $k+p$ for p, and (4)

$$kpw[s,u,t]/(k+p)(k+p-1)-ku[s,w,t]/(k+p)-pt[s,w,u]/(k+p),$$

for $-[(s, w, u)]$, and for $-[s, u, v]$,

$$kpv\,[sut]/(k+p)(k+p-1)-ku\,[svt]/(k+p)-pt\,[suv]/(k+p),$$

in (7), there results

$$\begin{aligned}&(\mathrm{a}\,...\between \partial_x\partial_y\partial_z)^2\,(tuvw)\\ &= (k+p+q+r-1)\,\big[vw\,(\mathrm{a}...\mathrm{f}...\between\partial_x\partial_y\partial_z)^2\,(tu)/(p+k-1)\\ &\qquad\qquad +qwut\,(\mathrm{a}a'+...)+ruvt\,(\mathrm{a}a''+...)\big]\\ &\quad -qrut\,[svw]-rpvt\,[swu]-pqwt\,[s, u, v]\\ &\quad +(krp+kpq)\,vw\,[sut]\,/\,(k+p-1)-kq\,[svt]-kr\,[swt].\end{aligned}$$

But (6) $(\mathrm{a}...\between\partial_x...)^2\,(tu)/(k+p-1) = pt\,(\mathrm{a}a+...)+ku\,(\mathrm{a}a'''+...)$
$$-kp\,[s, u, t]\,/\,(k+p-1),$$

substituting which in the preceding equation, the coefficient of $vw\,[s, u, t]$ becomes

$$\{kpr+kpq-(k+p+q+r-1)\,kp\}\,/\,(k+p-1) \quad \text{or} \quad kp,$$

and
$$\begin{aligned}&(\mathrm{a}...\mathrm{f}...\between\partial_x\partial_y\partial_z)^2\,(tuvw)\\ &= (k+p+q+r-1)\,\{kuvw\,(\mathrm{a}a'''+...+2\mathrm{f}f'''+...)\\ &\quad +p\,vwt\,(\mathrm{a}a+...)+q\,wtu\,(\mathrm{a}a'+...)+rtuv\,(\mathrm{a}a''+...)\}\\ &\quad -qrut\,[s, v, w]-rpvt\,[s, w, u]-pqwt\,[s, u, v]\\ &\quad -kpvw\,[s, u, t]-kqwu\,[s, v, t]-kruv\,[s, w, t] \quad ....(8).\end{aligned}$$

From (7), (8) the general law is evident, whatever be the number of forms in the product operated on; and similarly it may be shown that, if it holds for the product of $i-1$ functions, it will hold for that of i functions.

IV. *Results of operating on* vw, uvw ... *with*
$(\mathrm{abcfgh}\between n\partial_y-m\partial_z,\; l\partial_z-n\partial_x,\; m\partial_x-l\partial_y)^2$.

In the next place consider the operator formed by substituting in

$$s \equiv \mathrm{a}x^2+\mathrm{b}y^2+\mathrm{c}z^2+2\mathrm{f}yz+2\mathrm{g}zx+2\mathrm{h}xy,$$

$$n\frac{\partial}{\partial y}-m\frac{\partial}{\partial z} \text{ for } x,\; l\frac{\partial}{\partial z}-n\frac{\partial}{\partial x} \text{ for } y,\; m\frac{\partial}{\partial x}-l\frac{\partial}{\partial y} \text{ for } z;$$

it is evident that the result of operating with this will be deduced from the corresponding result of operating with the operator of the

first part of this paper by substituting in that result

$$\mathrm{b}n^2+\mathrm{c}m^2-2\mathrm{f}mn \text{ for a},\quad \mathrm{c}l^2+\mathrm{a}n^2-2\mathrm{g}nl \text{ for b},\quad \mathrm{a}m^2+\mathrm{b}l^2-2\mathrm{h}lm \text{ for c},$$

$$-\mathrm{a}mn-\mathrm{f}l^2+\mathrm{g}lm+\mathrm{h}nl \text{ for f},\quad -\mathrm{b}nl+\mathrm{f}lm-\mathrm{g}m^2+\mathrm{h}mn \text{ for g},$$

$$-\mathrm{c}lm+\mathrm{f}nl+\mathrm{g}mn-\mathrm{h}n^2 \text{ for h}.$$

Thus (4)

$$\left\{\mathrm{a}\left(n\frac{\partial}{\partial y}-m\frac{\partial}{\partial z}\right)^2+\ldots+2\mathrm{f}\left(l\frac{\partial}{\partial z}-n\frac{\partial}{\partial x}\right)\left(m\frac{\partial}{\partial x}-l\frac{\partial}{\partial y}\right)+\ldots\right\}(vw)$$

$$\begin{aligned}=(q+r-1)\Big[rv\big\{&(\mathrm{b}c''+\mathrm{c}b''-2\mathrm{f}f'')\,l^2+\ldots\\&\ldots+2\,(-\mathrm{a}f''-\mathrm{f}a''+\mathrm{g}h''+\mathrm{h}g'')\,mn+\ldots\big\}\\+qw\big\{&(\mathrm{b}c'+\mathrm{c}b'-2\mathrm{f}f')\,l^2+\ldots\big\}\Big]\end{aligned}$$

$$-qr\Sigma\left|\begin{aligned}&x^2\,\{B_1\,(\mathrm{b}l^2+\mathrm{a}m^2-2\mathrm{h}lm)+C_1\,(\mathrm{c}l^2+\mathrm{a}m^2-2\mathrm{g}nl)\\&\qquad\qquad+2F_1\,(\mathrm{a}mn+\mathrm{f}l^2-\mathrm{g}lm-\mathrm{h}nl)\}\\&+xy\,\{F_1(-\mathrm{b}nl+\mathrm{f}lm-\mathrm{g}m^2+\mathrm{h}mn)+G_1(-\mathrm{a}mn-\mathrm{f}l^2+\mathrm{g}lm+\mathrm{h}nl)\\&\qquad+C_1\,(\mathrm{c}lm-\mathrm{f}nl-\mathrm{g}mn+\mathrm{h}n^2)+H_1\,(-\mathrm{a}m^2-\mathrm{b}l^2+2\mathrm{h}lm)\}\\&+zx\,\{H_1(-\mathrm{a}mn-\ldots+\mathrm{h}nl)+F_1\,(-\mathrm{c}lm+\ldots-\mathrm{h}n^2)\\&\qquad+B_1\,(\mathrm{b}nl+\ldots+\mathrm{h}mn)+G_1\,(-\mathrm{c}l^2-\mathrm{a}n^2+2\mathrm{g}nl)\}\end{aligned}\right|$$

or, identically,

$$-qr\Sigma\left|\begin{aligned}&\left.\begin{aligned}&\mathrm{a}x^2\,(A_1l^2+\ldots+2F_1mn+\ldots)+l^2x^2\,(A_1\mathrm{a}+\ldots+2F_1\mathrm{f}+\ldots)\\&\quad-2\mathrm{a}xlx\,(A_1l+H_1m+G_1n)-2\mathrm{h}xlx\,(H_1l+B_1m+F_1n)\\&\quad-2\mathrm{g}xlx\,(G_1l+F_1m+C_1n)\end{aligned}\right\}\\&\left.\begin{aligned}&+\mathrm{h}xy\,(A_1l^2+\ldots+2F_1mn+\ldots)+lmxy\,(A_1\mathrm{a}+\ldots+2F_1\mathrm{f}+\ldots)\\&\quad-(\mathrm{a}xmy+\mathrm{h}ylx)(A_1l+H_1m+G_1n)\\&\quad-(\mathrm{h}xmy+\mathrm{b}ylx)(H_1l+B_1m+F_1n)\\&\quad-(\mathrm{g}xmy+\mathrm{f}ylx)(G_1l+F_1m+C_1n)\end{aligned}\right\}\\&\left.\begin{aligned}&+\mathrm{g}zx\,(A_1l^2+\ldots+2F_1mn+\ldots)+nlzx\,(A_1\mathrm{a}+\ldots+2F_1\mathrm{f}+\ldots)\\&\quad-(\mathrm{a}xnz+\mathrm{g}zlx)(A_1l+\ldots)-(\mathrm{h}xnz+\mathrm{f}zlx)(H_1l+\ldots)\\&\quad-(\mathrm{g}xnz+\mathrm{c}zlx)(G_1l+\ldots)\end{aligned}\right\}\end{aligned}\right|$$

or, if $\qquad s_1=\partial s/2\partial x\ldots,\quad L=lx+my+nz,$

$$-qr\Sigma\left|\begin{aligned}&s_1x\,(A_1l^2+\ldots+2F_1mn+\ldots)+Llx\,(A_1\mathrm{a}+\ldots+2F_1\mathrm{f}+\ldots)\\&\quad-(\mathrm{a}xL+s_1lx)(A_1l+\ldots)+(\mathrm{h}xL+s_2lx)(H_1l+\ldots)\\&\quad+(\mathrm{g}xL+s_3lx)(G_1l+\ldots)\end{aligned}\right|;$$

whence, finally,

$$\left\{\mathrm{a}\left(n\frac{\partial}{\partial y}-m\frac{\partial}{\partial z}\right)^2+\dots\right\}(vw)$$

$$=(q+r-1)\left[rv\left\{(\mathrm{bc}''+\mathrm{c}b''-2\mathrm{f}f'')\,l^2+\dots\right\}+qw\left\{(\mathrm{bc}'+\mathrm{c}b'-2\mathrm{f}f')\,l^2+\dots\right\}\right]$$
$$-qr\left[(A_1l^2+\dots+2F_1mn+\dots)\,s+(A_1\mathrm{a}+\dots+2F_1\mathrm{f}+\dots)\,L\right.$$
$$-2\left\{(A_1l+H_1m+G_1n)\,s_1+(H_1l+B_1m+F_1n)\,s_2\right.$$
$$\left.\left.+(G_1l+F_1m+C_1n)\,s_3\right\}L\right]\quad\dots\dots\dots\dots(9).$$

From the foregoing investigation it is evident that (7)

$$\left\{\mathrm{a}\left(n\frac{\partial}{\partial y}-m\frac{\partial}{\partial z}\right)^2+\dots\right\}(uvw)$$

$$(p+q+r-1)\left[pvw\left\{(\mathrm{bc}+\mathrm{c}b-2\mathrm{f}f)\,l^2+\dots\right\}+qwu\left\{(\mathrm{bc}'+\dots)\,l^2+\dots\right\}\right.$$
$$\left.+ruv\left\{(\mathrm{bc}''+\dots)\,l^2+\dots\right\}\right]$$
$$-\left\{qru\,(A_1l^2+\dots)+rpv\,(A_1'l^2+\dots)+pqw\,(A_1''l^2+\dots)\right\}s$$
$$-\left\{qru\,(A_1\mathrm{a}+\dots)+rpv\,(A_1'\mathrm{a}+\dots)+pqw\,(A_1''l^2+\dots)\right\}L^2$$
$$+2\left[\left\{qru\,(A_1l+H_1m+G_1n)+rpv\,(A_1'l+\dots)+pqw\,(A_1''l+\dots)\right\}s_1\right.$$
$$+\left\{qru\,(H_1l+B_1m+F_1n)+rpv\,(H_1'l+\dots)+pqw\,(H_1''l+\dots)\right\}s_2$$
$$\left.+\left\{qru\,(G_1l+F_1m+C_1n)+rpv\,(G_1'l+\dots)+pqw\,(G_1l+\dots)\right\}s_3\right]L$$
$$\dots\dots\dots\dots(10);$$

and that according to the same law the result of operating on the product of four, or more, functions may be at once written down.

A Theorem in Combinations. By R. W. D. Christie.

[*Read Jan. 10th,* 1889.]

Let $\Sigma\,({}_nS_r)^m$ represent the sum of the m^{th} powers of the sums of the letters which form the several combinations of n letters taken r at a time; then

$$\Sigma\,({}_nS_1)^m-\Sigma\,({}_nS_2)^m+\Sigma\,({}_nS_3)-\Sigma\,({}_nS_4)^m\,\dots\,\pm\,\Sigma\,({}_nS_n)^m\equiv 0,$$

where $m < n$, *e.g.*, if $n=5$, $m=4$,

$$a^4+b^4+c^4+d^4+e^4-(a+b)^4-(a+c)^4-(a+d)^4-(a+e)^4-(b+c)^4$$
$$-(b+d)^4-(b+e)^4-(c+d)^4-(c+e)^4-(d+e)^4+(a+b+c)^4$$

$$+(a+b+d)^4+(a+b+e)^4+(b+c+d)^4+(b+c+e)^4+(b+d+e)^4$$
$$+(c+d+e)^4+(a+c+d)^4+(a+c+e)^4+(a+d+e)^4-(a+b+c+d)^4$$
$$-(a+b+c+e)^4-(a+b+d+e)^4-(a+c+d+e)^4-(b+c+d+e)^4$$
$$+(a+b+c+d+e)^4 \equiv 0.$$

Here
$$\Sigma\,({}_5S_5)^4 = \Sigma a^4+4\Sigma a^3b+6\Sigma a^2b^2+12\Sigma a^2bc+24\Sigma abcd$$
$$\Sigma\,({}_5S_4)^4 = 4\Sigma a^4+12\Sigma a^3b+18\Sigma a^2b^2+24\Sigma a^2bc+24\Sigma abcd$$
$$\Sigma\,({}_5S_3)^4 = 6\Sigma a^4+12\Sigma a^3b+18\Sigma a^2b^2+12\Sigma a^2bc$$
$$\Sigma\,({}_5S_2)^4 = 4\Sigma a^4+4\Sigma a^3b+6\Sigma a^2b^2$$
$$\Sigma\,({}_5S_1)^4 = \Sigma a^4.$$

Thus the number of times that

Σa^4 or a^4, b^4, c^4 ... will occur will be ${}_4C_0-{}_4C_1+{}_4C_2-{}_4C_3+{}_4C_4 = (1-1)^4=0$,

$4\Sigma a^3b$ or $4a^3b$, $4a^3c$ or $6\Sigma a^2b^2$ or $6a^2c^2$ ... $-{}_3C_0+{}_3C_1-{}_3C_2+{}_3C_3 = -(1-1)^3=0$,

$12\Sigma a^2bc$ .... ${}_2C_0-{}_2C_1+{}_2C_2 = (1-1)^2=0$,

$24\Sigma abcd$ .... $-{}_1C_0+{}_1C_1 = -(1-1)^1=0$,

and there is no term containing the five letters.

In the general case, the number of times that a^m, b^m, c^m ... will occur, will be

$${}_{n-1}C_0-{}_{n-1}C_1+{}_{n-1}C_2- \dots {}_{n-1}C_{n-1} = (1-1)^{n-1} = 0;$$

that $\frac{m\,!}{p\,!\;q\,!\;r\,!}\,a^p\,b^q\,c^r$ ... will occur, will be

$$(-1)^{k-1}\left[{}_{n-k}C_0-{}_{n-k}C_1+\dots\right] = (-1)^{k-1}(1-1)^{n-k} = 0,$$

where $p+q+r \dots = m$, and the letters $a, b, c \dots$ are k in number.

The general explanation is, that the number of ways of selecting r things out of n so that each selection shall contain k given things is

$${}_{n-k}C_{r-k}.$$

The theorem will enable us to obtain an unlimited supply of equations such as the following:—

Let $\quad a=1,\ b=2,\ c=3,\ d=5,\ e=7,\ f=8.$

Then, after cancelling terms like C^m with $-(a+b)^m$, &c., we get

$$0^m+4^m+9^m+17^m+22^m+26^m \equiv 1^m+2^m+12^m+14^m+24^m+25^m,$$

where $m = 1, 2, 3, 4,$ or 5; and therefore for the same magnitudes

of m

$$n^m+(n+4)^m+(n+9)^m+(n+17)^m+(n+22)^m+(n+26)^m$$
$$\equiv(n+1)^m+(n+2)^m+(n+12)^m+(n+14)^m+(n+24)^m+(n+25)^m$$

for all magnitudes of n.

Thursday, February 14th, 1889.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Mr. Baker was admitted into the Society.

The following communications were made:—

On the Diophantine Relation $y^2+y'^2=$ square: Prof. Cayley, F.R.S.

Sur la Transformation des Équations Algébriques: Signor Brioschi.

On Projective Cyclic Concomitants or Surface Differential Invariants: E. B. Elliott, M.A.

On Secondary Invariants: Prof. L. J. Rogers, M.A.

Remarks upon Algebraical Symmetry, with particular reference to the Theory of Operations, and the Theory of Distributions: Major Macmahon, R.A.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLV., Nos. 273 and 274.

"Educational Times," for February.

"Greek Geometry, from Thales to Euclid," by Dr. G. J. Allman, 8vo; Dublin and London, 1889.

"Bulletin des Sciences Mathématiques," November and December, 1888, and January, 1889.

"Beiblätter zu den Annalen der Physik und Chemie," Band XII., Stück 12; Band XIII., Stück 1.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," No. 73; Index to Ditto, for 1888 (part only).

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo II., No. 5; Mexico, 1888.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. IV., Fasc. 6, 7, 8, 9.

"Annales de l'Ecole Polytechnique de Delft," Tome IV., Livraison 3me; Leide, 1888.

"Annali di Matematica," Tome XVI., Fasc. 3.

"Journal für die reine und angewandte Mathematik," Band 104, Heft. II.

"Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften," Band 14, Nos. 10, 11, 12, 13; Leipzig, 1888.

On the Diophantine Relation, $y^2+y'^2 =$ *Square.* *By* Prof. CAYLEY.

[*Read Feb.* 14*th*, 1889.]

The diophantine relation $y^2+y'^2 =$ Square, where y is a function of x, and y' denotes $\frac{dy}{dx}$, is considered by Prof. Sylvester in his paper "On Reducible Cyclodes," *Proc. Lond. Math. Soc.*, t. I. (1865–66), pp. 137–160. It is at once seen that there exists a solution

$$y = (x+a)^{\alpha}(x+b)^{\beta}(x+c)^{\gamma}(x+d)^{\delta}\ldots,$$

where the roots $a, b, c, d, \ldots$ are essentially unequal, and the number of simple factors $x+a, x+b, x+c, x+d, \ldots$ is even; the exponents $\alpha, \beta, \gamma, \delta, \ldots$ are taken to be positive integer numbers. Sylvester assumes, and it will be shown, that the factors must separate themselves into two sets, or, as he calls them, diptychs, each containing the same number of simple factors, and such that the sum of the exponents for the one diptych is equal to the sum of the exponents for the other diptych; viz., the form is $y = UU_1$, where

$$U = (x+a)^{\alpha}(x+b)^{\beta}\ldots,\quad U_1 = (x+a_1)^{\alpha_1}(x+b_1)^{\beta_1}\ldots,$$

with the same number of simple factors, and with the relation $\alpha+\beta+\ldots = \alpha_1+\beta_1+\ldots$ between the exponents. Hence, if the number of simple factors be called the class and the sum of the exponents be called the order, the class and the order are each of them even; or, what is the same thing, the semi-class (say μ) and the semi-order (say ν) are each of them integral.

The separation of the factors into two diptychs is a remarkable theorem. I consider the analytical theory; for greater simplicity, first in the case, class $= 2$, and secondly in the case, class $= 4$; but it is easy to see that the like process is applicable to the case of any even value whatever of the class.

I write as usual $i = \sqrt{-1}$; the equation $y^2+y'^2 =$ square, implies $y+iy' =$ square, and $y-iy' =$ square; at least this is so, save as to a common denominator, as will appear.

First, if the class is $= 2$; we have

$$y = (x+a)^{\alpha}(x+b)^{\beta};$$

hence
$$y+iy' = y\left(1+\frac{2\alpha}{x+a}+\frac{2\beta}{x+b}\right),$$
$$y-iy' = y\left(1-\frac{2\alpha}{x+a}-\frac{2\beta}{x+b}\right),$$
say these are $=\dfrac{(x+l)^2}{x+a\,.\,x+b}$ and $\dfrac{(x+m)^2}{x+a\,.\,x+b}$ respectively;

and, this being so, we have
$$y^2+y'^2 = \frac{y^2\,(x+l)^2\,(x+m)^2}{(x+a)^2\,(x+b)^2}, \quad = (x+a)^{2\alpha-2}(x+b)^{2\beta-2}(x+l)^2(x+m)^2.$$

It is to be shown that the assumed relations lead to $\alpha = \beta$. Resolving the last mentioned expressions for $y+iy'$, $y-iy'$ each into simple fractions, we have
$$i\alpha\,(b-a) = (l-b)^2, \quad -i\alpha\,(b-a) = (m-b)^2,$$
$$i\beta\,(a-b) = (l-a)^2, \quad -i\beta\,(a-b) = (m-a)^2.$$
Hence $\quad (l-b)^2+(m-b)^2 = 0, \quad (l-a)^2+(m-a)^2 = 0;$

these cannot give
$$(l-b)+i\,(m-b) = 0, \quad (l-a)+i\,(m-a) = 0,$$
with the same sign for i in the two equations; for we should then have
$$(1+i)(b-a) = 0,$$
but $1+i$ is not $= 0$, and a, b are essentially unequal. Hence, taking (as we may do) $+i$ in the first equation, we must have $-i$ in the second equation, and the two equations will be
$$l-b+i\,(m-b) = 0, \quad \text{that is} \quad l+im = (1+i)\,b,$$
$$l-a-i\,(m-a) = 0, \quad \text{,,} \quad l-im = (1-i)\,a,$$
and thence
$$2l = (1+i)(b-ia),$$
$$2m = (1+i)(b+ia),$$
Hence also
$$2\,(l-b) = (1-i)(a-b), \quad 2i\,(m-b) = (1-i)(b-a),$$
$$2\,(l-a) = -(1+i)(a-b), \quad 2i\,(m-a) = (1+i)(b-a);$$
consequently,
$$2\,(l-b)^2 = -i\,(a-b)^2, = -2i\alpha\,(a-b),$$
$$2\,(l-a)^2 = i\,(a-b)^2, = 2i\beta\,(a-b).$$
Hence $\alpha = \beta = \frac{1}{2}\,(a-b)$, and the solution thus is
$$y = (x+a)^\alpha\,(x+b)^\alpha, \qquad \alpha = \tfrac{1}{2}\,(a-b),$$

$$l = \tfrac{1}{2}\left\{a+b+i(a-b)\right\},$$
$$m = \tfrac{1}{2}\left\{a+b-i(a-b)\right\},$$
$$y^2+y'^2 = (x+a)^{2a-2}(x+b)^{2a-2}(x+l)^2(x+m)^2.$$

The class is here $= 2$, and the order is $= 2a$; considering the order as given, say it is$=2\nu$, we have $a = \nu$, and the equation $\nu = \frac{1}{2}(a-b)$ then shows that one of the roots a, b is arbitrary. Taking it to be a, we have $b = a-2\nu$, or the solution, class 2 and order 2ν, is

$$y = (x+a)^{\nu}(x+a-2\nu)^{\nu},$$
$$l = a-\nu+i\nu, \qquad m = a-\nu-i\nu,$$
$$y^2+y'^2 = (x+a)^{2\nu-2}(x+a-2\nu)^{2\nu-2}(x+l)^2(x+m)^2.$$

Considering next for the case, class $= 4$, the solution

$$y = (x+a)^{\alpha}(x+b)^{\beta}(x+c)^{\gamma}(x+d)^{\delta},$$

we have $$y+iy' = y\left(1+\frac{i\alpha}{x+a}+\frac{i\beta}{x+b}+\frac{i\gamma}{x+c}+\frac{i\delta}{x+d}\right),$$
$$y-iy' = y\left(1-\frac{i\alpha}{x+a}-\frac{i\beta}{x+b}-\frac{i\gamma}{x+c}-\frac{i\delta}{x+d}\right);$$

or, putting these

$$= \frac{(x+l)^2(x+p)^2}{x+a\,.\,x+b\,.\,x+c\,.\,x+d} \text{ and } \frac{(x+m)^2(x+q)^2}{x+a\,.\,x+b\,.\,x+c\,.\,x+d} \text{ respectively,}$$

we have

$$y^2+y'^2 = \frac{y^2(x+l)^2(x+p)^2(x+m)^2(x+q)^2}{(x+a)^2(x+b)^2(x+c)^2(x+d)^2},$$
$$=(x+a)^{2\alpha-2}(x+b)^{2\beta-2}(x+c)^{2\gamma-2}(x+d)^{2\delta-2}(x+l)^2(x+p)^2(x+m)^2(x+q)^2.$$

Also, by decomposing the expressions for $y+2y'$, $y-2y'$ into simple fractions and comparing with the original values, we find

$$i\alpha(b-a)(c-a)(d-a) = (a-l)^2(a-p)^2,$$
$$i\beta(a-b)(c-b)(d-b) = (b-l)^2(b-p)^2,$$
$$i\gamma(a-c)(b-c)(d-c) = (c-l)^2(c-p)^2,$$
$$i\delta(a-d)(b-d)(c-d) = (d-l)^2(d-p)^2,$$

$$-i\alpha(b-a)(c-a)(d-a) = (a-m)^2(a-q)^2,$$
$$-i\beta(a-b)(c-b)(d-b) = (b-m)^2(b-q)^2,$$
$$-i\gamma(a-c)(b-c)(d-c) = (c-m)^2(c-q)^2,$$
$$-i\delta(a-d)(b-d)(c-d) = (d-m)^2(d-q)^2,$$

Hence
$$(a-l)^2(a-p)^2+(a-m)^2(a-q)^2=0,$$
$$(b-l)^2(b-p)^2+(b-m)^2(b-q)^2=0,$$
$$(c-l)^2(c-p)^2+(c-m)^2(c-q)^2=0,$$
$$(d-l)^2(d-p)^2+(d-m)^2(d-q)^2=0,$$
and we cannot from these obtain *three* equations
$$(a-m)(a-q)-i(a-l)(a-p)=0,$$
$$(b-m)(b-q)-i(b-l)(b-p)=0,$$
$$(c-m)(c-q)-i(c-l)(c-p)=0,$$
with the same sign for i; in fact these would give
$$(1+i)(b-c)(c-a)(a-b)=0,$$
but $1+i$ is not $=0$, and the a, b, c are essentially unequal. Hence we must have equations such as
$$(a-m)(a-q)-i(a-l)(a-p)=0;\ (c-m)(c-q)+i(c-l)(c-p)=0,$$
$$(b-m)(b-q)-i(b-l)(b-p)=0;\ (d-m)(d-q)+i(d-l)(d-p)=0,$$
two of them with $-i$, and two of them with $+i$; viz., the a, b, c, d divide themselves into pairs which are taken to be a, b and c, d.

We hence easily obtain,
$$a+b-m-q-i(a+b-l-p)=0,\quad ab-mq-i(ab-lp)=0,$$
$$c+d-m-q-i(c+d-l-p)=0,\quad cd-mq-i(cd-lp)=0,$$
and thence
$$a+b-c-d=i(a+b+c+d)-2i(l+p),$$
$$ab-cd=i(ab+cd)\qquad -2ilp.$$

Forming from these values of $l+p$, lp the expression for $2i(a-l)(a-p)$, we find $2i(a-l)(a-p)=(i+1)(a-c)(a-d)$; and we have thus the set of equations
$$2i(a-l)(a-p)=(i+1)(a-c)(a-d),$$
$$2i(b-l)(b-p)=(i+1)(b-c)(b-d),$$
$$2i(c-l)(c-p)=(i-1)(c-d)(c-b),$$
$$2i(d-l)(d-p)=(i-1)(d-a)(d-b).$$

Hence also
$$\begin{aligned}
2(a-l)^2(a-p)^2 &= -i(a-c)^2(a-d)^2,\\
2(b-l)^2(b-p)^2 &= -i(b-c)^2(b-d)^2,\\
2(c-l)^2(c-p)^2 &= i(c-a)^2(c-b)^2,\\
2(d-l)^2(d-p)^2 &= i(d-a)^2(d-b)^2;
\end{aligned}$$
and, substituting these values in a former set of equations, we obtain
$$\begin{aligned}
2\alpha(b-a) &= -(a-c)(a-d),\\
2\beta(a-b) &= -(b-c)(b-d),\\
2\gamma(d-c) &= (c-a)(c-b),\\
2\delta(c-d) &= (d-a)(d-b);
\end{aligned}$$
and thence
$$\begin{aligned}
2(\alpha+\beta) &= a+b-c-d,\\
2(\gamma+\delta) &= -(c+d-a-b);
\end{aligned}$$
that is, $\alpha+\beta=\gamma+\delta$; viz., there are, in this case also, two diptychs.

If, as before, the order is taken to be $=2\nu$, then $\alpha+\beta=\nu$, $\gamma+\delta=\nu$; supposing that ν is a given positive integer, and that α, β, γ, δ are positive integers satisfying these equations $\alpha+\beta=\nu$, $\gamma+\delta=\nu$, then the last-mentioned four equations between α, β, γ, δ and a, b, c, d are equivalent to three relations serving to determine the differences of a, b, c, d (say $a-d$, $b-d$, $c-d$) in terms of α, β, γ, δ. And we then further have
$$\begin{aligned}
(a-l)(a-p) &= -(1-i)\,\alpha(b-a), & (a-m)(a-q) &= -(1+i)\,\alpha(b-a),\\
(b-l)(b-p) &= -(1-i)\,\beta(a-b), & (b-m)(b-q) &= -(1+i)\,\beta(a-b),\\
(c-l)(c-p) &= (1+i)\,\gamma(d-c), & (c-m)(c-q) &= (1-i)\,\gamma(c-d),\\
(d-l)(d-p) &= (1+i)\,\delta(c-d), & (d-m)(d-q) &= (1-i)\,\delta(d-c),
\end{aligned}$$
each set equivalent to two equations; or, as these may be written,
$$\begin{aligned}
2(l+p) &= a+b+c+d+i(a+b-c-d),\\
2lp &= ab+cd \quad +i(ab-cd),\\
2(m+q) &= a+b+c+d-i(a+b-c-d),\\
2mq &= ab+cd \quad -i(ab-cd),
\end{aligned}$$
serving to determine l, p, m, q in terms of a, b, c, d.

Observe also that, u being arbitrary, we have
$$\begin{aligned}
2(u-l)(u-p) &= (1+i)(u-a)(u-b)+(1-i)(u-c)(u-d),\\
2(u-m)(u-q) &= (1-i)(u-a)(u-b)+(1+i)(u-c)(u-d),
\end{aligned}$$
(which equations, writing therein $u=a$, b, c, or d, in fact reproduce the two systems of four equations).

We have also

$$l+p+m+q = a+b+c+d, \quad l+p-m-q = i\,(a+b-c-d),$$
$$lp+mq = ab+cd, \quad lp-mq = i\,(ab-cd)\,;$$

and moreover

$$4\,(l-p)^2 = 2i\,\{(a-b)^2-(c-d)^2\}+4\,(a+b)(c+d)-8\,(ab+cd),$$
$$4\,(m-q)^2 = -2i\,\{(a-b)^2-(c-d)^2\}+4\,(a+b)(c+d)-8\,(ab+cd),$$

which equations, combined with the foregoing values of $2\,(l+p)$ and $2\,(m+q)$, give the values of l, p, m, q. We have thus the complete solution for the case class $= 4$, order $= 20$; say

$$y = (x+a)^{\alpha}\,(x+b)^{\beta}.\,(x+c)^{\gamma}\,(x+d)^{\delta}\,; \quad \alpha+\beta = \gamma+\delta = \nu,$$
$$y^2+y'^2 = (x+a)^{2\alpha-2}\,(x+b)^{2\beta-2}\,(x+c)^{2\gamma-2}\,(x+d)^{2\delta-2}$$
$$\times\,(x+l)^2\,(x+p)^2\,(x+m)^2\,(x+q)^2,$$

with the foregoing relations between the constants.

Sur la Transformation des Équations Algébriques.

By Signor Brioschi.

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1. Soit $f(x) = 0$ une équation du dégré n et $x_0, x_1 \ldots x_{n-1}$ ses racines.

Si l'on pose: $\quad (1)\ldots\ldots f(x) = (x-x_r)\,\phi\,(x),$

étant x_r une quelconque des racines, et:

$$f'(x_0)\,f'(x_1)\ldots f'(x_{n-1}) = \Delta_f,$$

on trouve que: $\quad \Delta_f = (-1)^{n-1}\,\phi^2\,(x_r)\,\Delta_{\phi},$

lequel résultat peut s'exprimer comme il suit. L'invariant (discriminant) Δ_f du degré $2\,(n-1)$ de la forme f est égal à un covariant de l'ordre $2\,(n-1)$, et du degré $2\,(n-1)$ de la forme ϕ.

Or un théorème analogue se vérifie pour chaque covariant et pour chaque invariant de la forme f, comme je vais démontrer. Soit F un covariant de la forme f, covariant de l'ordre m et du degré p. Le

covariant F doit satisfaire les trois équations différentielles :

$$\sum_1^n {}_s\, s f_{s-1} \frac{dF}{df_s} = 0, \quad \sum_0^n {}_s\, f_s \frac{dF}{df_s} = pF,$$

$$\sum_1^n {}_s\, s f_s \frac{dF}{df_s} = \tfrac{1}{2}(np-m)\, F,$$

dans lesquelles :

$$f_0 = f(x), \quad f_1 = \frac{1}{n} f'(x), \quad f_2 = \frac{1}{n(n-1)} f''(x), \ldots\ldots$$

Mais, en posant :

$$\phi_0 = \phi(x_r), \quad \phi_1 = \frac{1}{n-1}\phi'(x_r), \quad \phi_2 = \frac{1}{(n-1)(n-2)}\phi''(x_r), \ldots\ldots$$

de la relation (1) entre f et ϕ, on déduit

$$f_0(x_r) = 0, \quad f_s(x_r) = \frac{s}{n}\phi_{s-1},$$

et en substituant ces valeurs de $f_0, f_1 \ldots$ dans le covariant F, les trois équations différentielles supérieures se transforment dans les suivantes :

$$(2) \ldots\ldots \sum_1^{n-1} {}_s\, s\phi_{s-1} \frac{dF}{d\phi_s} = 0, \quad \sum_0^{n-1} {}_s\, \phi_s \frac{dF}{d\phi_s} = pF,$$

$$\sum_0^{n-1} {}_s\, (s+1)\,\phi_s \frac{dF}{d\phi_s} = \tfrac{1}{2}(np-m)\, F,$$

ou en retranchant de celle-ci la seconde :

$$(3) \ldots\ldots \sum_1^{n-1} {}_s\, s\phi_s \frac{dF}{d\phi_s} = \tfrac{1}{2}\left[(n-1)p-(m+p)\right] F.$$

Les trois équations (2), (3) démontrent que le covariant F de l'ordre m et du degré p de la forme f dans lequel on pose pour x une quelconque x_r des racines de l'équation $f(x) = 0$, peut s'exprimer par un covariant de la forme ϕ, covariant de l'ordre $m+p$ et du degré p. Si F est un invariant de la forme f, l'on a $m = 0$, et on retombe dans le théorème démontré pour le discriminant.

2. Je vais donner un exemple de ces relations en me limitant pour le moment au cas de $n = 5$.

La forme ϕ a trois covariants :

$$\phi, \quad h = \tfrac{1}{2}(\phi\phi)_2, \quad t = 2(\phi h),$$

et deux invariants g_2, g_3, liées entre eux par la relation :

$$t^2 = -4h^3 + g_2 h\phi^2 - g_3\phi^3.$$

La forme f a dix-neuf covariants, quatre invariants, mais pour le but que j'ai en vue, il me suffit de considérer les quatre covariants linéaires, les trois quadratiques, les trois cubiques, et les invariants. En posant :

$$l = \tfrac{1}{2}(ff)_4, \quad p = -\tfrac{1}{3}(fl)_2,$$
$$m = \tfrac{1}{2}(pp)_2, \quad n = (lm), \quad q = (lp), \quad r = (mp),$$
$$\alpha = (lp)_2, \quad \beta = (l\alpha), \quad \gamma = (m\alpha), \quad \delta = (n\alpha),$$
$$A = \tfrac{1}{2}(ll)_2, \quad B = (lm)_2, \quad C = \tfrac{1}{2}(mm)_2, \quad D = (\alpha\delta),$$

on obtient

1. Les quatre invariants A, B, C, D des degrés 4, 8, 12, 18.
2. Les quatre covariants linéaires α, β, γ, δ des degrés 5, 7, 11, 13.
3. Les trois covariants quadratiques l, m, n des degrés 2, 6, 8.
4. Les trois covariants cubiques p, q, r des degrés 3, 5, 9.

En conséquence du théorème démontré au s° 1r, les covariants l, m, n sont des covariants de ϕ des degrés 2, 6, 8 et des ordres 4, 8, 10. On trouve en effet :

$$l = -\frac{3 \cdot 4}{5^2} \cdot h, \quad m = \frac{1}{3^2 \cdot 5^6}\left[144 g_3 \phi h + 96 g_2 h^2 - 25 g_2^2 \phi^2\right],$$
$$n = \frac{2}{3 \cdot 5^7}\left[144 g_3 h - 5 g_2^2 \phi\right] t.$$

Les trois covariants p, q, r s'expriment en fonctions de covariants de ϕ des degrés 3, 5, 9 et des ordres 6, 8, 12 ; et une calculation très-facile donne :

$$p = -\frac{4}{5^3}\, t, \quad q = \frac{4}{5^5}(6 g_3 \phi - g_2 h)\, \phi,$$

$$3^3 \cdot 5^9 \cdot r = (5^4 \cdot g_2^3 + 2 \cdot 3^3 \cdot 4^3 \cdot g_3^2)\, \phi^3 - 8 \cdot 9\left[66 g_2 g_3 \phi^2 + 7 g_2^2 h \phi - 240 g_3 h^2\right] h.$$

On aura de même :

$$\alpha = -\frac{8}{5^4}\, g_2 t, \quad \beta = \frac{4}{5^6}\left[12 g_2 g_3 \phi^2 - 7 g_2^2 h \phi + 144 g_3 h^2\right],$$
$$\gamma = \frac{2}{5}\, g_2 r - \frac{1}{3 \cdot 5^4}(144 g_3 h - 5 g_2^2 \phi)\, m,$$

dans laquelle on doit poser pour r, m les valeurs supérieures, et enfin :

$$\delta = \frac{2}{3 \cdot 5^{11}}\, t\left[288 g_2^2 g_3 \phi h - 128\,(g_2^3 + 54 g_3^2)\, h^2 + 25 g_2^4 \phi^2\right].$$

On voit que quatre de ces covariants α, δ, n, p ont pour facteur le covariant gauche t; les autres sont fonctions de ϕ, h, g_2, g_3.

Quant aux invariants on trouve directement:

$$A = \frac{12}{5^4}\left[3g_3\phi - 8g_2 h\right],$$

et l'on déduit B de la relation entre les discriminants:

$$16\,(g_2^3 - 27g^2)\,\phi^2 = 5^5\,(A^2 - 144B).$$

La valeur de C peut s'obtenir directement, ou plus simplement de la manière suivante:—En posant:

$$L = 4(2AB - 27C),$$

on a, comme il est connu, la relation:

$$Ll = \beta^2 + A\alpha^2,$$

le second membre de laquelle, en substituant les valeurs de α, β, A, est divisible par h ou par l, et l'on déduit ainsi la valeur de C.

3. Entre les covariants considérés dans le s. précédent le seul q jouit de la propriété d'avoir comme facteur ϕ; mais on voit tout-de-suite qu'on peut par des combinaisons des autres covariants obtenir des expressions qui aient la même propriété.

Une première combinaison est la suivante:

$$(4)\ \ldots\ldots\ 54r + l\beta,$$

qui donne un covariant de f de troisième ordre et du neuvième degré; une seconde et une troisième, les

$$(5)\ \ldots\ldots\ 5Ar + 4n\alpha, \quad 25A^2r + 8\alpha^2\beta,$$

d'ordre 3 et des degrés 13, 17.

Évidemment chacune de ces quatre expressions étant divisibles par ϕ, en se rappellant que $\phi = f'(x_r)$, donnent quatre formules de transformation de l'équation $f(x) = 0$, et dans chacun des quatre cas la transformée a le coefficient du second terme égal à zero, et les coefficients des termes suivants seront des fonctions entières de A, B, C.

En posant: $$y = 10\,\frac{q}{\phi} = 10\,\frac{q\,(x_r)}{f'\,(x_r)}$$

les valeurs de q, A données au s. précédent, conduisent aux relations suivantes: $$(6)\ \ldots\ldots\ g_2 h = \frac{5^3}{2.4.3^2}\,(3y - 4A), \quad g_3\phi = \frac{5^3}{4.3^3}\,(12y - A),$$

et de la rèlation entre les discriminants on obtient, à cause de cette dernière: (7) ...... $g_2^3\phi^2 = \frac{5^6}{4^3 . 3^3}\left[5(12y-A)^2+27(A^2-144B)\right]$.

Enfin de la valeur de L on déduit

$$(8) \ldots\ldots g_2^2 h^3 = -\frac{5^{10}}{4^6 . 3^7}\Big[870y^3+165Ay^2-40A^2y-4050By+\tfrac{16}{9}A^3 \\ +216AB-4^2 . 3^7 . C\Big].$$

Au moyen de ces quatre relations on arrive à exprimer les (4), (5) en fonction de y, A, B, C. Mais auparavant il importe d'observer qu'en multipliant les deux dernières (7), (8) entre elles, on a, à cause des (6), que le premier membre est une fonction de y et de A ; une très-simple calculation conduit à l'équation :

$$y^5-10By^3-40Cy^2+5(5B^2+\tfrac{4}{3}AC)y-(\tfrac{16}{9}A^2C+\tfrac{4}{3}AB^2-216BC) = 0,$$

transformée en y de l'équation $f(x) = 0$.

On trouve pour le covariant (4) :

$$(10) \ldots\ldots \frac{54r+l\beta}{f'(x)} = 9y^2+4Ay-36B,$$

et analoguement on aura pour les covariants (5) deux polynomes du troisième et du quatrième degré en y.

En multipliant y et les trois polynomes en y des degrés 2, 3, 4 par des indeterminées, et en nommant avec z leur somme, par l'expression en z, analogue à celle de Tchirnauss, on transformera l'équation $f(x) = 0$ dans une autre, pour laquelle le coefficient du second terme est égal à zéro et les coefficients suivants seront fonctions de A, B, C et des quatre indéterminées.

Je démontrérais dans une prochaine occasion l'application de la méthode exposée à la transformation des équations du septième degré.

On Projective Cyclic Concomitants, or Surface Differential Invariants. *By* E. B. Elliott, M.A.

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Contents.

I.

1. In analogy with Professor Sylvester's nomenclature in connection with Reciprocants, I propose to give the name Projective or Principiant Cyclicants to those Pure Cyclicants* which are Differential, Invariants for all homographic transformations of the three variables of the second and higher derivatives of one of which with regard to the other two they are functions, *i.e.*, to those which have the property of persistence in form, but for a factor not involving second and higher derivatives, when the variables undergo any such transformation as

$$\frac{x}{a_1x'+b_1y'+c_1z'+d_1}=\frac{y}{a_2x'+b_2y'+c_2z'+d_2}=\frac{z}{a_3x'+b_3y'+c_3z'+d_3}$$
$$=\frac{1}{Ax'+By'+Cz'+D} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

It will be remembered that all pure cyclicants persist for the included most general *linear* transformation, in which the A, B, C of (1) are zeroes.

Projective cyclicants obey of course all the laws of pure cyclicants in general; *i.e.*, they are homogeneous, doubly isobaric, and subject to annihilation by the four operators (see *Proceedings*, Vol. xviii., pp. 142, 164).

$$\Omega_1\equiv\Sigma\left\{(m+1)\,x_{m+1,n-1}\frac{d}{dx_{mn}}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

* *Cf. Proceedings*, Vol. xix., pp. 377, &c.

$$\Omega_2 \equiv \Sigma \left\{ (n+1)\, x_{m-1,\, n+1} \frac{d}{dx_{mn}} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)},$$

$$V_1 \equiv \Sigma \left\{ \Sigma \left(r x_{rs} x_{m+1-r,\, n-s} \right) \frac{d}{dx_{mn}} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(4)},$$

$$V_2 \equiv \Sigma \left\{ \Sigma \left(s x_{rs} x_{m-r,\, n+1-s} \right) \frac{d}{dx_{mn}} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(5)},$$

in which x_{mn} denotes $\dfrac{1}{m!\ n!} \dfrac{d^{m+n} x}{dy^m\, dz^n}$ for all values of m, n, and in which the limits of the summations are adequately expressed by saying that in every derivative z_{pq} which occurs, whether in a coefficient or in an operating symbol, p and q must be positive integers, or a positive integer and a zero, whose sum is not less than 2. We shall presently see that the further conditions, necessary and sufficient, for a pure cyclicant to be projective are, that it have the two additional annihilators

$$\omega_1 \equiv \Sigma \left\{ (m+n-2)\, x_{m,\, n-1} \frac{d}{dx_{mn}} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(6)},$$

$$\omega_2 \equiv \Sigma \left\{ (m+n-2)\, x_{m-1,\, n} \frac{d}{dx_{mn}} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(7)}.$$

It will be noticed that I am, for subsequent convenience, using a notation which regards y and z as the independent variables, and x as the dependent. In passages where this is not the case, Ω_1, Ω_2, V_1, V_2, ω_1, ω_2 will still be used to denote the operators (2) to (7) with the dependent variable, whichever it may be, written in them for x.

2. The formulæ of transformation (1) may be replaced by the succession of the three following substitutions

$$\left.\begin{aligned} Cx &= (Ca_1 - c_1 A)\, X + (Cb_1 - c_1 B)\, Y + (Cd_1 - c_1 D)\, Z + Cc_1 \\ Cy &= (Ca_2 - c_2 A)\, X + (Cb_2 - c_2 B)\, Y + (Cd_2 - c_2 D)\, Z + Cc_2 \\ Cz &= (Ca_3 - c_3 A)\, X + (Cb_3 - c_3 B)\, Y + (Cd_3 - c_3 D)\, Z + Cc_3 \end{aligned}\right\} \ldots \text{(8)},$$

$$\frac{X}{X'} = \frac{Y}{Y'} = \frac{Z}{1} = \frac{1}{Z'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(9)},$$

$$\left.\begin{aligned} X' &= x' \\ Y' &= \qquad y' \\ CZ' &= Ax' + By' + Cz' + D \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(10)},$$

the only case of failure being when $C = 0$. In this special case A and B cannot be both also zero without the transformation (1)

degenerating into a merely linear one. Suppose then that B, for instance, is different from zero. The transformation (1) may now be effected by the series of transformations (8), (9), (10), altered only by the interchange of C, c_1, c_2, c_3, and B, b_1, b_2, b_3, preceded by the particular linear substitution of x for y and y for x, and followed by that of y' for x' and x' for y'. In all cases, therefore, the homographic transformation (1) may be replaced by a succession of linear transformations and a transformation like (9).

II.

3. We have, accordingly, to study the transformation (9), or say

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{1} = \frac{1}{z'} \quad \text{.......................} \quad (11),$$

with a view to determine differential expressions which persist in form after the transformation, and in particular cyclicants which have this property of persistence. There are advantages of simplicity, as the sequel will make sufficiently clear, in regarding y and z as the independent variables, so that the relation, of course perfectly unrestricted in form, which is supposed to connect x, y, z, is regarded as one expressing the first in terms of the second and third of these variables. Similarly, of x', y', z', the two last are taken as the independent variables. In this and the following nine articles, forms of persistent expressions for the transformation (11) are investigated without any special reference to the theory of cyclicants.

A reason for the greater simplicity gained by regarding x and x' as the dependent variables in the two sets is, that the formulæ for the transformation of the independent variables

$$y = \frac{y'}{z'}, \quad z = \frac{1}{z'} \quad \text{.......................} \quad (12),$$

or

$$y' = \frac{y}{z}, \quad z' = \frac{1}{z} \quad \text{.......................} \quad (13),$$

are thus quite unencumbered by any presence of the dependent. Thus we obtain from them at once the equivalences of operators

$$\frac{d}{dy} = \frac{1}{z}\frac{d}{dy'} = z'\frac{d}{dy'} \quad \text{.......................} (14),$$

$$\frac{d}{dz} = -\frac{y}{z^2}\frac{d}{dy'} - \frac{1}{z^2}\frac{d}{dz'} = -y'z'\frac{d}{dy'} - z'^2\frac{d}{dz'} \quad \text{.........} (15),$$

from which also

$$-yz\frac{d}{dy} - z^2\frac{d}{dz} = \frac{d}{dz'} \text{.......................} (16).$$

The completely reciprocal relations between the accented and unaccented letters in (11), and its consequences, are of fundamental importance.

It is to be remarked that the operative symbols in (14), (15), (16) are symbols of total and not partial differentiation, so that, for instance, if the function operated on involve the dependent variable x or x' explicitly, $\frac{d}{dy}$ stands for $\left[\frac{d}{dy}\right]+x_{10}\left(\frac{d}{dx}\right)$ in which $\left(\frac{d}{dx}\right)$ is the symbol of differentiation with regard to x in so far as it is explicitly involved, and $\left[\frac{d}{dy}\right]$ that for all those parts of the operation $\frac{d}{dy}$ which ignore the explicit presence of x. We shall be in little danger of confusion in this matter, for it is only at the outset that we shall have occasion to operate on functions in which the dependent variable explicitly appears.

It is easy from (14)—(16) to obtain two independent linear differential operators which persist in form after the transformation. From (16), by aid of (12),

$$z^{\frac{1}{2}}\frac{d}{dy} = z'^{\frac{1}{2}}\frac{d}{dy'} \quad \text{..............................(17)},$$

and by subtraction of (16) from z^2 times (15), and use of (12),

$$y\frac{d}{dy}+2z\frac{d}{dz} = -\left\{y'\frac{d}{dy'}+2z'\frac{d}{dz'}\right\} \quad \text{..................(18)}.$$

It is clear, then, that we are to expect two classes of persistent functions—a class which persist absolutely, and a skew class which persist but for a change of sign. They may be called persistents of positive and negative character respectively, and the operators (17) and (18) may be called, respectively, positively and negatively persistent operators.

Other persistent operators, equivalent of course in the aggregate to these two only, may be with ease written down. Thus the sum and difference of (15) and (16) give us, respectively, the positively and negatively persistent operators

$$yz\frac{d}{dy}+(z^2-1)\frac{d}{dz} = y'z'\frac{d}{dy'}+(z'^2-1)\frac{d}{dz'} \quad \text{............(19)},$$

$$yz\frac{d}{dy}+(z^2+1)\frac{d}{dz} = -\left\{y'z'\frac{d}{dy'}+(z'^2+1)\frac{d}{dz'}\right\} \quad \text{......(20)};$$

and, again, the sum and difference of (16) and z times (15) give us

$$z^{\frac12}\left\{y\frac{d}{dy}+(z-1)\frac{d}{dz}\right\}=z'^{\frac12}\left\{y'\frac{d}{dy'}+(z'-1)\frac{d}{dz'}\right\} \quad \text{......(21)},$$

and
$$z^{\frac12}\left\{y\frac{d}{dy}+(z+1)\frac{d}{dz}\right\}=-z'^{\frac12}\left\{y'\frac{d}{dy'}+(z'+1)\frac{d}{dz'}\right\} \quad \text{......(22)}.$$

It should be remarked that in (17), and either (19) or (21), we have two independent positively persistent operators.

4. The persistent operators arrived at in the last article enable us at once to write down any number of persistent functions of the variables and derivatives, when we notice that the formulæ of transformation (11) may themselves be written in persistent form. Thus, in the independent variable z only, we have the persistents, not of course independent,

$$z+\frac{1}{z}=z+\frac{1}{z'} \quad \text{..........................(23)},$$

$$z^{-\frac12}(z+1)=z'^{-\frac12}(z'+1) \quad \text{....................(24)},$$

$$z^{-\frac12}(z-1)=-z'^{-\frac12}(z'-1) \quad \text{....................(25)},$$

$$\log z=-\log z'=w, \text{ say} \quad \text{....................(26)},$$

&c., &c. The two first of these are of positive, and the third and fourth of negative, character.

Again, in both independent variables y and z, we have the persistent

$$z^{-\frac12}y=z'^{-\frac12}y'=v, \text{ say} \quad \text{....................(27)},$$

and in the dependent and independent variables

$$z^{-\frac12}x=z'^{-\frac12}x'=u, \text{ say} \quad \text{......................(28)},$$

or, again,
$$y^{-1}x=y'^{-1}x' \quad \text{............................(29)}.$$

In (26), (27), and (28), we have in persistent form the exact equivalents of (11).

By operation on (28) or (29) with any one of the persistent operators (17) to (22) we get a linear persistent function in x, x_{10}, x_{01}, or some of them, the coefficients involving one or both of y and z. By a second operation with the same or another of the operators, we get a new persistent linear in x, x_{10}, x_{01}, x_{20}, x_{11}, x_{02}, or some of them. In fact, we get a linear persistent after any number of repetitions of such operations. Let us choose two independent persistent operators, (17) and (18) say. By use of them as thus indicated, noticing that

the effect of the compound operation

$$\left(y\frac{d}{dy}+2z\frac{d}{dz}\right)\left(z^{\frac{1}{2}}\frac{d}{dy}\right)$$

is not altered by reversing the order of its component simple operations, we obtain a perfectly complete system of linear persistent functions by assigning to m and n, in

$$\left(z^{\frac{1}{2}}\frac{d}{dy}\right)^m\left(y\frac{d}{dy}+2z\frac{d}{dz}\right)^n(z^{-\frac{1}{2}}x)$$

$$=(-1)^n\left(z'^{\frac{1}{2}}\frac{d}{dy'}\right)^m\left(y'\frac{d}{dy'}+2z'\frac{d}{dz'}\right)^n(z'^{-\frac{1}{2}}x')\text{......}(30),$$

all zero and positive integral values in succession. The functions are positively or negatively persistent as n is even or odd.

For the explanation of (30) we have not far to seek. The relation, whatever it may be, which connects x, y, z, may be expressed in terms of the elementary persistent functions u, v, w, of (28), (27), and (26). Thus we have

$$\left.\begin{aligned} z &= e^{w} \\ y &= ve^{\frac{1}{2}w} \\ x &= ue^{\frac{1}{2}w} \end{aligned}\right\}\text{..............................(31)},$$

and the companion formulæ

$$\left.\begin{aligned} z' &= e^{-w} \\ y' &= ve^{-\frac{1}{2}w} \\ x' &= ue^{-\frac{1}{2}w} \end{aligned}\right\}\text{..............................(32)}.$$

The first two of each of these groups give us

$$\frac{d}{dv}=e^{\frac{1}{2}w}\frac{d}{dy}=e^{-\frac{1}{2}w}\frac{d}{dy'}$$

$$=z^{\frac{1}{2}}\frac{d}{dy}=z'^{\frac{1}{2}}\frac{d}{dy'}\text{ ....(33)},$$

and

$$\frac{d}{dw}=e^{w}\frac{d}{dz}+\tfrac{1}{2}ve^{\frac{1}{2}w}\frac{d}{dy}=-e^{-w}\frac{d}{dz'}-\tfrac{1}{2}ve^{-\frac{1}{2}w}\frac{d}{dy'}$$

$$=\tfrac{1}{2}\left(y\frac{d}{dy}+2z\frac{d}{dz}\right)=-\tfrac{1}{2}\left(y'\frac{d}{dy'}+2z'\frac{d}{dz'}\right)\text{ ....(34)}.$$

Thus the two sides of (30) are merely equivalent expressions for

$$2^n \frac{d^{m+n} u}{dv^m dw^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35),$$

and the completeness of the series of linear persistent expressions given by (30) or (35) lies in the fact that these are constant multiples of the entire system of coefficients in the expansion of an increment of u or $z^{-\frac{1}{2}}x$ in terms of increments of v or $z^{-\frac{1}{2}}y$ and w or $\log z$.

It is at once clear that a complete system of persistent functions, with y instead of x taken for dependent variable, is afforded in like manner by the series of derivatives

$$\frac{d^{m+n} v}{dw^m du^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36).$$

The case when z is taken as the dependent variable has less simplicity, for u, v, and w all involve z, while only one of them involves x or y.

5. It is not to be assumed that the complete system of linear persistents for the transformation (11) given by (30) or (35) is the simplest in form of all complete systems when written explicitly. This statement may be illustrated by writing down the complete system for the first two orders.

Operating on (28) with (17) and (18) in turn, we get the complete system of the first order

$$\frac{du}{dv} = x_{10} = x'_{10} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37),$$

$$2\frac{du}{dw} = z^{-\frac{1}{2}}(yx_{10} + 2zx_{01} - x) = -z'^{-\frac{1}{2}}(y'x'_{10} + 2z'x'_{01} - x') \quad \ldots (38).$$

An equivalent pair, obtained by operating on (28) with (21) and (22), and remembering (24) and (25), is

$$yx_{10} + (z-1)\,x_{01} - x = y'x'_{10} + (z'-1)\,x'_{01} - x' \quad \ldots\ldots\ldots\ldots (39),$$

$$yx_{10} + (z+1)\,x_{01} - x = -\left\{y'x'_{10} + (z'+1)\,x'_{01} - x'\right\} \quad \ldots\ldots (40).$$

Note that the formulæ giving x_{10} and x_{01} in terms of accented letters are (37), and

$$x_{01} = x' - y'x'_{10} - z'x'_{01} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (41).$$

The complete set of linear persistents of the second order are

$$\tfrac{1}{2}\frac{d^2u}{dv^2} = z^{\frac{1}{2}}x_{20} = z'^{\frac{1}{2}}x'_{20} \dots\dots\dots\dots (42),$$

$$\frac{d^2u}{dv\,dw} = yx_{20}+zx_{11} = -(y'x'_{20}+z'x'_{11}) \dots\dots\dots\dots (43),$$

and
$$2^{\frac{1}{2}}\frac{d^2u}{dw^2} = 2z^{-\frac{1}{2}}y^2x_{20}+4z^{\frac{1}{2}}yx_{11}+8z^{\frac{3}{2}}x_{02}-z^{-\frac{1}{2}}y\,x_{10}+z^{-\frac{1}{2}}x$$
$$= +\text{(the same expression in accented letters)} \dots (44).$$

Now, the last two terms in (44) are themselves positively persistent functions, by (27), (37), and (28). So too is its first term, by (42) and (27). Thus our simplest complete system of the second order consists of (42), (43), and the remaining terms of (44), *i.e.*,

$$z^{\frac{1}{2}}(yx_{11}+2zx_{02}) = z'^{\frac{1}{2}}(y'x'_{11}+2z'x'_{02}) \dots\dots\dots\dots (45).$$

There are advantages, however, as will be seen presently, in not omitting the first term of (44), but in taking, rather than (45) the somewhat less simple persistent

$$z^{-\frac{1}{2}}(y^2x_{20}+2zyx_{11}+4z^2x_{02}) = z'^{-\frac{1}{2}}(y'^2x'_{20}+2z'y'x'_{11}+4z'^2x'_{02}) \dots (46).$$

Again, note that the formulæ for x_{20}, x_{11}, x_{02} in terms of accented letters are

$$\left.\begin{aligned} x_{20} &= z'x'_{20} \\ x_{11} &= -z'(2y'x'_{20}+z'x'_{11}) \\ x_{02} &= z'(y'^2x'_{20}+y'z'x'_{11}+z'^2x'_{02}) \end{aligned}\right\} \dots\dots\dots\dots (47).$$

6. In the last article we have found a complete system of three linear persistents of the second order, which do not involve the dependent variable x nor the first derivatives x_{10}, x_{01}. Now the persistent operators (33) and (34) cannot produce functions involving x, x_{10}, x_{01} from functions which are free from them. It follows that, from the second order onwards, a complete system exists which only involves the independent variables y, z, and second and higher derivatives. It will now be proved that the type of such a complete system is

$$u_{mn} = z^{\frac{1}{2}(2n+m-1)}\,e^{\frac{1}{2}z^{-1}(\omega_1+y\Omega_1)}\,x_{mn} \dots\dots\dots\dots (48),$$

where $m+n \nless 2$, and ω_1, Ω_1 denote the operators (6) and (2).

We have already a complete system of the second order which

accords with the type in (42), (43), (46), *i.e.*, in

$$u_{20} = z^{\frac{1}{2}} x_{20} \quad \text{...} (42),$$

$$u_{11} = z x_{11} + y x_{20}$$

$$= z \left\{ 1 + \frac{y}{2z} 2x_{20} \frac{d}{dx_{11}} \right\} x_{11} \quad \text{...} (43a),$$

$$u_{02} = z^{-\frac{1}{2}} \left\{ z^2 x_{02} + \tfrac{1}{2} z y x_{11} + \tfrac{1}{4} y^2 x_{20} \right\}$$

$$= z^{\frac{1}{2}} \left\{ 1 + \frac{y}{2z} x_{11} \frac{d}{dx_{02}} + \frac{1}{1.2} \left(\frac{y}{2z} 2x_{20} \frac{d}{dx_{11}} \right) \left(\frac{y}{2z} x_{11} \frac{d}{dx_{02}} \right) \right\} x_{02} \ldots (46a),$$

of which the first two are $\frac{1}{2!} \frac{d^2u}{dv^2}$, $\frac{1}{1!\,1!} \frac{d^2u}{dv\,dw}$, and the third differs from $\frac{1}{2!} \frac{d^2u}{dw^2}$ by a persistent of the same character. The general law will be proved to be that u_{m0} and u_{m1} are $\frac{1}{m!} \frac{d^m u}{dv^m}$ and $\frac{1}{m!\,1!} \frac{d^{m+1}u}{dv^m\,dw}$, while, in general, u_{mn} differs from $\frac{1}{m!\,n!} \frac{d^{m+n}u}{dv^m\,dw^n}$ by a persistent of the same character not involving x_{mn}.

In the first place, that u_{m0} is the persistent $\frac{1}{m!} \frac{d^m u}{dv^m}$ is proved instantaneously as follows

$$\frac{1}{m!} \frac{d^m u}{dv^m} = \frac{1}{m!} \left(z^{\frac{1}{2}} \frac{d}{dy} \right)^m (z^{-\frac{1}{2}} x) = z^{\frac{1}{2}(m-1)} x_{m0}$$

$$= z^{\frac{1}{2}(m-1)} e^{\frac{1}{2}z^{-1}(\omega_1 + y\Omega_1)} x_{m0}$$

$$= u_{m0} \quad \text{..} (49).$$

We proceed to ground a mathematical induction, proving that u_{mn} is in all cases a persistent function, upon the actual evaluation of $\frac{d}{dw} u_{mn}$.

7. The order of the operations of ω_1 and Ω_1 on any function of the derivatives is immaterial, for it is at once clear that the alternant identity

$$\omega_1 \Omega_1 - \Omega_1 \omega_1 = 0$$

is satisfied.

Thus (48) may be written

$$u_{mn} = z^{\frac{1}{2}(2n+m-1)} \Sigma \left(\frac{1}{r!\,s!} \frac{y^s}{(2z)^{r+s}} \omega_1^r \Omega_1 \right) x_{mn}$$

$$= \Sigma \left(y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{1}{2^{r+s} r!\,s!} \omega_1^r \Omega_1^s \right) x_{mn}$$

$$= \Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} (m+n-2)(m+n-3)\ldots \right.$$
$$\left. \ldots(m+n-r-1)(m+1)(m+2)\ldots(m+s)\, x_{m+s,\, n-r-s} \right\}$$
$$= \Sigma \left(y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s)!}{2^{r+s} r!\, s!\, m!\,(m+n-r-2)!} x_{m+s,\, n-r-s} \right) \ldots (50),$$

the summation being with regard to r and s, and comprising all pairs of values (including zero) of those numbers whose sum does not exceed n.

It follows that

$$\frac{d}{dw} u_{mn} = \left(z \frac{d}{dz} + \tfrac{1}{2} y \frac{d}{dy} \right) u_{mn}$$

$$= \Sigma \left\{ y^s z^{\frac{1}{2}(2n+m+1-2r-2s)} \frac{(m+n-2)!\,(m+s)!\,(n-r-s+1)}{2^{r+s} r!\, s!\, m!\,(m+n-r-2)!} x_{m+s,\, n-r-s+1} \right\}$$
$$+ \Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(2n+m-1-2r-2s)(m+n-2)!\,(m+s)!}{2^{r+s+1} r!\, s!\, m!\,(m+n-r-2)!} \times x_{m+s,\, n-r-s} \right\}$$
$$+ \Sigma \left\{ y^{s+1} z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s+1)!}{2^{r+s+1} r!\, s!\, m!\,(m+n-r-2)!} x_{m+s+1,\, n-r-s} \right\}$$
$$+ \Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s)!}{2^{r+s+1} r!\,(s-1)!\, m!\,(m+n-r-2)!} x_{m+s,\, n-r-s} \right\};$$

in which the first summation may, by putting $r+1$ for r, be written

$$\Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s)!\,(n-r-s)}{2^{r+s+1}(r+1)!\, s!\, m!\,(m+n-r-3)!} x_{m+s,\, n-r-s} \right\},$$

and the third, by putting $r+1$ for r and $s-1$ for s,

$$\Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s)!}{2^{r+s+1}(r+1)!\,(s-1)!\, m!\,(m+n-r-3)!} x_{m+s,\, n-r-s} \right\}.$$

In these two summations the value -1 of r must be regarded as admissible, and in the latter the value 0 of s is excluded. Subject to this remark we obtain, then,

$$\frac{d}{dw} u_{mn} = \Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s)!}{2^{r+s+1}(r+1)!\, s!\, m!\,(m+n-r-2)!} x_{m+s,\, n-r-s} \right.$$
$$\left. \begin{aligned} \big[&(n-r-s)(m+n-r-2) \\ &+(2n+m-1-2r-2s)(r+1) \\ &+s\,(m+n-r-2) \\ &+(r+1)\,s \big] \end{aligned} \right\} \ldots\ldots\ldots (51),$$

the value -1 of r being admitted in the first and third products within the square brackets. It is also clear that the admission of the same value into the second and fourth products will introduce only zero terms to the summation, since $r+1$ is a factor of each of those terms, and since $(r+1)\,!$ in the denominator is taken as unity when $r+1$ vanishes. Again, that the value $s=0$ is excluded from the third product makes no difference, for a like reason.

Now, it is readily seen that the sum of products in the square brackets is

$$(n+1)(m+n-1)-(r+1)(r+s)$$
$$=(n+1)(m+n-1)-r\,(r+1)-s\,(r+1)\ldots\ldots\ldots\ldots(52).$$

It will be convenient to put r for $r+1$, since this may have all values. Doing so, we find that

$$\frac{d}{dw}u_{mn}=(n+1)\,\Sigma\left\{y^s z^{\frac12(2n+m+1-2r-2s)}\right.$$
$$\left.\times\frac{(m+n-1)\,!\,(m+s)\,!}{2^{r+s}\,r\,!\,s!\,m\,!\,(m+n-r-1)\,!}\,x_{m+s,\,n+1-r-s}\right\}$$
$$-\frac{(m+n-1)(m+n-2)}{2^2}\,\Sigma\left\{y^s z^{\frac12(2n'+m-1-2r'-2s)}\right.$$
$$\left.\times\frac{(m+n'-2)\,!\,(m+s)\,!}{2^{r'+s}\,r'\,!\,s\,!\,m\,!\,(m+n'-r'-2)\,!}\,x_{m+s,\,n'-r'-s}\right\}$$
$$-\frac{y}{z^{\frac12}}\,\frac{(m+1)(m+n-1)}{2^2}\,\Sigma\left\{y^{s''} z^{\frac12(2n''+m''-1-2r''-2s'')}\right.$$
$$\left.\times\frac{(m''+n''-2)\,!\,(m''+s'')\,!}{2^{r''+s''}\,r''\,!\,s''\,!\,m''\,!\,(m''+n''-r''-2)\,!}\,x_{m''+s'',\,n''-r''-s''}\right\}\ldots(53),$$

where

$$n'=n-1,\quad r'=r-2,\quad m''=m+1,\quad n''=n-1,\quad r''=r-1,\quad s''=s-1.$$

Consequently, by (50),

$$\frac{d}{dw}u_{mn}=(n+1)\,z^{\frac12(2\overline{n+1}+m-1)}\,e^{\frac12 z^{-1}(\omega_1+y\Omega_1)}\,x_{m,\,n+1}$$
$$-\frac{(m+n-1)(m+n-2)}{2^2}\,z^{\frac12(2\overline{n-1}+m-1)}\,e^{\frac12 z^{-1}(\omega_1+y\Omega_1)}\,x_{m,\,n-1}$$
$$-\frac{y}{z^{\frac12}}\,\frac{(m+1)(m+n-1)}{2^2}\,z^{\frac12(2\overline{n-1}+\overline{m+1}-1)}\,e^{\frac12 z^{-1}(\omega_1+y\Omega_1)}\,x_{m+1,\,n-1}$$
$$=u_{m,\,n+1}-\tfrac14\,(m+n-1)\,(m+n-2)\,u_{m,\,n-1}$$
$$-\tfrac14 yz^{-\frac12}\,(m+1)\,(m+n-1)\,u_{m+1,\,n-1}\ldots\ldots\ldots(54).$$

It would, at first sight, appear as if the second and third members of the right-hand side of (54) are not complete, but need to be reinforced by the addition of terms corresponding to the values -2, -1 for r' in the second, and -1 for r'' and -1 for s'' in the third member on the right of (53). But this is not the case. Values -2, -1 of r' would mean values $-1, 0$ of r in (51) and (52), for which the term $r(r+1)$ in (52) vanishes. Again, -1 for r'' and -1 for s'' would mean 0 for s and -1 for r, respectively, in (51) and (52), values which make $s(r+1)$ vanish.

A special result of greater simplicity replaces (54) for all cases when $n = 0$, the then meaningless symbols $u_{m,\,n-1}$ and $u_{m+1,\,n-1}$ having in such cases to be replaced by zeroes. For, when $n = 0$, s and r can be only zero in (50), and consequently in (52) s can only be zero and r only zero or -1; so that the first of the three parts of the right-hand member of (53) or (54) is the only one that exists.

This is easy to see by actual operation on u_{m0} without introduction of the general notation. Thus

$$\frac{d}{dw}u_{m0} = \left(z\frac{d}{dz}+\tfrac{1}{2}y\frac{d}{dy}\right)\left(z^{\frac{1}{2}(m-1)}x_{m0}\right)$$

$$= z^{\frac{1}{2}(m+1)}x_{m1}+\tfrac{1}{2}(m-1)\,z^{\frac{1}{2}(m-1)}x_{m0}+\tfrac{1}{2}yz^{\frac{1}{2}(m-1)}(m+1)\,x_{m+1,0}$$

$$= z^{\frac{1}{2}(m+1)}\left\{1+\frac{1}{2z}\omega_1+\frac{y}{2z}\Omega_1\right\}x_m$$

$$= u_{m1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(55).$$

8. The materials for a mathematical induction proving u_{mn} a persistent for the transformation (11), whatever numbers (including zero) m and n be, provided that $m+n \nless 2$, are now ready. By (49) u_{m0} is always a persistent. By (55) so is u_{m1}. Now (54) tells us that, if u_{mn}, $u_{m,\,n-1}$ and $u_{m+1,\,n-1}$ are persistents, $u_{m,\,n+1}$ must be one. Thus, since u_{m1}, u_{m0}, $u_{m+1,0}$ are persistents, so is u_{m2}; since u_{m2}, u_{m1}, $u_{m+1,1}$ are, so is u_{m3}; since u_{m3}, u_{m2}, $u_{m+1,2}$ are, so is u_{m4}, &c. Thus, finally, u_{mn} is one for all values of n as well as for all of m.

That the general persistent u_{mn} is linear, is clear from the method of its formation, ω_1 and Ω_1 being lineo-linear operators. That all the linear persistents u_{mn} are independent is also evident, for in the order u_{20}, u_{11}, u_{02}, u_{30}, u_{21}, u_{12}, u_{03}, u_{40}, u_{31}, &c., each involves one derivative of x which does not appear in any of the preceding. That the system is complete from the second order onwards follows from the fact that up to any order it has just as many members as there are of second and

higher derivatives $\dfrac{d^{m+n}u}{dv^m\,dw^n}$ or x_{mn}. With u, $\dfrac{du}{dv}$, and $\dfrac{du}{dw}$, [see (28), (37), and (38)], the system is absolutely complete.

9. The absolutely complete system of linear persistents for the transformation (11), which is now before us, suffices of course to produce by combinations of its members every persistent which exists, non-linear as well as linear. We now enter upon the theory of the formation of non-linear persistents.

It is well known that, if ϑ be any linear differential operator, and U, V any two functions of the arguments on which it operates, and if ϑ be called ϑ_1 when it and its repetitions act on U only, and ϑ_2 when upon V only, then

$$\vartheta^n(UV) = (\vartheta_1+\vartheta_2)^n(UV),$$

for all positive integral values of n, and consequently

$$\begin{aligned} e^{\vartheta}(uv) &= e^{\vartheta_1+\vartheta_2}(uv) \\ &= e^{\vartheta_1}.\,e^{\vartheta_2}(uv) \\ &= e^{\vartheta_1}u\,.\,e^{\vartheta_2}v \\ &= e^{\vartheta}u\,.\,e^{\vartheta}v \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(56). \end{aligned}$$

Now $\dfrac{1}{2z}(\omega_1+y\Omega_1)$ is such an operator ϑ upon functions of the second and higher derivatives x_{mn}. It follows, from (56) and (48), that

$$\begin{aligned} u_{mn}u_{m'n'} &= z^{\frac12(2n+m-1)}\,e^{\frac12 z^{-1}(\omega_1+y\Omega_1)}\,x_{mn}\,z^{\frac12(2n'+m'-1)}\,e^{\frac12 z^{-1}(\omega_1+y\Omega_1)}\,x_{m'n'} \\ &= z^{\frac12(2n+2n'+m+m'-2)}\,e^{\frac12 z^{-1}(\omega_1+y\Omega_1)}\,(x_{mn}x_{m'n'}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(57). \end{aligned}$$

The right-hand member of this equality is then a persistent.

The extensions to products of 3, 4, ..., and finally all numbers of elementary persistents u_{mn}, is effected in like manner, the general conclusion being that, $\Pi^{(i)}u_{mn}$ denoting a product of i factors u_{mn},

$$\Pi^{(i)}u_{mn} = z^{\frac12(2\Sigma n+\Sigma m-i)}\,e^{\frac12 z^{-1}(\omega_1+y\Omega_1)}\,(\Pi^{(i)}x_{mn}) \quad \ldots\ldots\ldots(58).$$

Consequently, taking any sum of a number of such products of i factors for all of which Σm and Σn are constant numbers, w_1 and w_2 respectively, we deduce that, if $H^{(i)}_{w_1 w_2}(u)$ denote a homogeneous (of

degree i) and doubly isobaric (of partial weights w_1 and w_2) function of the second and higher linear persistents u_{mn}, and if $H^{i}_{w_1, w_2}(x)$ denote the same function of the corresponding derivatives x_{mn}, then

$$H^{(i)}_{w_1, w_2}(u) = z^{\frac{1}{2}(2w_2+w_1-i)} e^{\frac{1}{2}z^{-1}(x_{1}+y\Omega_1)i} H^{i}_{w_1, w_2}(x) \quad \text{...............(59).}$$

The right-hand member is then a persistent. Its sign character will be + or − according as w_2 is even or odd. Thus the identity expressive of its persistency is

$$z^{\frac{1}{2}(2w_2+w_1-i)} e^{\frac{1}{2}z^{-1}(x_1+y\Omega_1)i} H^{i}_{w_1, w_2}(x) = (-1)^{w_2} z'^{\frac{1}{2}(2w_2+w_1-i)} e^{\frac{1}{2}z'^{-1}(x'_1+y'\Omega'_1)i} H^{i}_{w_1, w_2}(x')$$

$$\text{............(60).}$$

10. The fundamental linear persistents u_{20}, u_{11}, u_{02} ... u_{mn}, ... are free from the dependent variable x and the first derivatives x_{10} and x_{01}. The same will consequently be the case with the rational integral persistents of any degree given by (59). On the other hand, the independent variables y and z occur in all the linear persistents, except that y is absent when $n = 0$. As a rule, therefore, both y and z will occur in all the rational and integral persistents (59).

Now y and z occur in different manners in u_{mn}. This linear persistent contains terms free from y and terms involving $y, y^2 \ldots y^n$ respectively as factors. The leading term $z^{\frac{1}{2}(2n+m-1)} x_{mn}$ is among those which have no power of y as a factor. On the other hand, this leading term has for a factor the other independent variable z raised to a power which is never zero but always positive. There is in fact no term in the expression for u_{mn} which has not a positive power of z for a factor, except when $m = 0$, in which case a single term has for its z factor the negative power $z^{-\frac{1}{2}}$. Moreover, no other term involves so high a power of z as the leading one.

We cannot expect, then, to find rational integral persistents (59) which do not involve z. It is now to be seen that there are, however, a vast, and no doubt infinite, number which only involve a single power of z as a factor throughout, and which do not involve y at all. Such will be called rational integral *pure* persistents. By dividing one pure persistent by a suitable power of any other, the z factor may be made to disappear, thus yielding us an *absolute* pure persistent. Absolute pure persistents are of necessity fractional.

Now, in (59), the leading term $H^{i}_{w_1, w_2}(x)$ is the one on the right which certainly cannot be made to disappear; for, if it did, so would $H^{(i)}_{w_1, w_2}(u)$, and (59) would be a mere identity of zeroes. We seek then, in order to make H a pure persistent, necessary and sufficient

conditions that all the other terms disappear. These conditions are at once seen to be

$$\omega_1 H^{(i)}_{w_1 w_2}(x) = 0, \text{ and } \Omega_1 H^{(i)}_{w_1 w_2}(x) = 0 \ldots\ldots\ldots\ldots (61).$$

The two are necessary, for the coefficients of the various powers and products of powers of the independent quantities z and y must vanish separately, and consequently, in particular, the coefficients of $z^{\frac{1}{2}(2w_2+w_1-i-2)}$ and $yz^{\frac{1}{2}(2w_2+w_1-i-2)}$ must vanish. They are also sufficient,

for, if $$\omega_1 H = 0, \text{ and } \Omega_1 H = 0,$$

it will be a consequence that

$$\omega_1^r H = 0, \quad \omega_1^r \Omega_1^s H = 0, \text{ and } \Omega_1^s H = 0 \ldots\ldots\ldots\ldots (62),$$

for all positive integral values of r and s.

Space will not now permit a systematic classification of the pure persistents for the transformation (11) to which we thus obtain the clue, nor a development of their interesting properties. It will be remembered that we have been only seeking them in order to discuss the selection from them of those which are also cyclicants, *i.e.*, persistents, but for a first derivative factor, for linear transformations of the variables. In the rest of this paper they will then be dealt with only in their bearing on the theory of cyclicants and kindred functions.

11. It is worth while to remark that I first arrived at the necessary and sufficient conditions (61), that a pure function H be, but for a power of z as factor, a persistent for the transformation (11), in a somewhat different manner, by proving that, since

$$x' = z^{-1} x,$$

$$\frac{d}{dy'} = z \frac{d}{dy},$$

and
$$\frac{d}{dz'} = -yz \frac{d}{dy} - z^2 \frac{d}{dz},$$

$$(-1)^n x'_{mn} = z^{2n+m-1} e^{(1/z)(\omega_1 + y\Omega_1)} x_{mn}\text{*} \ldots\ldots\ldots\ldots\ldots (63),$$

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* Comparison of (63) with (48) gives us the following rule for deducing u_{mn}, or rather $(-1)^n 2^{\frac{1}{2}(2n+m-1)} u_{mn}$, from the value of x'_{mn} in terms of variables and unaccented derivatives:—"Divide by the square root of the multiplier of x_{mn}, and in the quotient replace z by $2z$."

For the deduction of x'_{mn} from u_{mn} the rule is—

"Multiply by $(-1)^n$ and by $(4z)^{\frac{1}{2}(2n+m-1)}$, and then replace z by $\frac{1}{2}z$."

and consequently that

$$(-1)^{w_2} H^{(i)}_{w_1 w_2}(x') = z^{2w_2+w_1-i}\, e^{(1/z)(\omega_1+y\Omega_1)}\, H^{(i)}_{w_1 w_2}(x) \quad \text{.........} \quad (64).$$

The method is on the whole easier than that above detailed, for, in the work corresponding to that of Art. 7 above, no residual terms such as the $(r+1)(r+s)$ of (52) occur. The advantage of the method which I have here followed lies in its incidental investigation of complete systems of linear and rational integral persistents which are not pure or free from y.

Combination of the two results (60) and (64) leads to an interesting conclusion which might undoubtedly be proved independently and made the basis of a third method. By operating on each side of (64) with $e^{(-1/2z)(\omega_1+y\Omega_1)}$, and multiplying by $z'^{\frac{1}{2}(2w_2+w_1-i)}$, we get

$$(-1)^{w_2} z'^{\frac{1}{2}(2w_2+w_1-i)}\, e^{(-1/2z)(\omega_1+y\Omega_1)}\, H^{(i)}_{w_1 w_2}(x') = z^{\frac{1}{2}(2w_2+w_1-i)}\, e^{(1/2z)(\omega_1+y\Omega_1)}\, H^{(i)}_{w_1 w_2}(x).$$

But, by (60), the right-hand member of this identity is also equal to

$$(-1)^{w_2} z'^{\frac{1}{2}(2w_2+w_1-i)}\, e^{(1/2z')(\omega_1'+y'\Omega_1')}\, H^{(i)}_{w_1 w_2}(x').$$

We have then the identity of operators

$$e^{(-1/2z)(\omega_1+y\Omega_1)} = e^{(1/2z')(\omega_1'+y'\Omega_1')} \quad \text{.....................} \quad (65),$$

the function operated on upon the left being any function of the second and higher derivatives of x with regard to y and z, or, more generally, of any function of those derivatives and of y and z, but not also of x, x_{10}, x_{01}. The transform of such a function is necessarily free from x', x'_{10}, x'_{01}.

12. The results as to the transformation (11) with y instead of x taken as dependent variable are, from the complete symmetry of the formulæ of transformation in x and y, at once seen to be deduced from those already obtained by mere interchange of the letters x and y and of first and second suffixes throughout.

For the purposes of the rest of this paper, it is best to state the equivalent of this fact in a somewhat different way. Companion to the transformation (11), *i.e.*,

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{1} = \frac{1}{z'} \quad \text{.....................} \quad (11),$$

we have the transformation

$$\frac{x}{x''} = \frac{y}{1} = \frac{1}{y''} = \frac{z}{z''} \quad \text{.......................} \quad (66).$$

For this transformation as for the other there is a complete system of linear persistents of which the type, for values of m and n whose sum exceeds unity, is

$$y^{\frac{1}{2}(2m+n-1)}\, e^{(1/2y)(\omega_2+z\Omega_2)}\, x_{mn} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (67),$$

and a complete system of rational integral persistents

$$y^{\frac{1}{2}(2w_1+w_2-i)}\, e^{(1/2y)(\omega_2+z\Omega_2)}\, H^{(i)}_{w_1 w_2}(x) \ldots\ldots\ldots\ldots\ldots \quad (68),$$

in which ω_2, Ω_2 are the operators (7) and (3), obtained by interchanging first and second suffixes in ω_1 and Ω_1 respectively. The necessary and sufficient conditions that the homogeneous and doubly isobaric function H be a pure persistent for the transformation (66) are then

$$\omega_2 H = 0 \text{ and } \Omega_2 H = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (69).$$

III.

13. We now proceed to the consideration of homogeneous and doubly isobaric functions of the second and higher derivatives, which, as well as being annihilated by one or both of the operators ω_1, ω_2, and one or both of Ω_1, Ω_2, have the further property of being annihilated by one or both of the operators V_1, V_2. Three classes of these functions will occupy our attention. The most restricted class is mentioned first.

A. *Projective or Principiant Cyclicants*, as defined in Art. 1. It is now seen that the necessary and sufficient conditions which they satisfy are that they have all six annihilators ω_1, ω_2, Ω_1, Ω_2, V_1, V_2. In the next article but one, it will be made clear that their annihilation by ω_2 and V_2 is a mere consequence of their annihilation by the other four.

B. *Projective Semicyclicants.*—These possess the three annihilators ω_1, Ω_1, V_1. Their property is that they persist in form, but for a factor involving the variables and first derivatives, after any transformation which can be produced by a succession of transformations like (11), and special linear transformations like

$$\left.\begin{aligned} x &= lX + mY + nZ + p \\ y &= l'X + m'Y + n'Z + p' \\ z &= \qquad\qquad\quad n''Z + p'' \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (70),$$

i.e., after any special homographic transformation, such as

$$\frac{x}{ax'+by'+cz'+d} = \frac{y}{a'x'+b'y'+c'z'+d'} = \frac{z}{c''z'+d''} = \frac{1}{Cz'+D} \ldots (71).$$

C. ***Principiant Semicyclicants***, the specially interesting class of projective semicyclicants which have the fourth annihilator ω_2 as well as the three ω_1, Ω_1, V_1.

Some projective and principiant semicyclicants are annihilated by V_2 as well as by the operators, annihilation by which defines them, and have in consequence additional properties. It is probably unwise, however, to burden the subject with further nomenclature.

14. The following important alternant identities are easily verified :—

$$\omega_1\omega_2-\omega_2\omega_1 = 0 \quad\text{..........}(72),$$

$$\omega_1\Omega_1-\Omega_1\omega_1 = 0 \quad\text{..........}(73),$$

$$\omega_2\Omega_2-\Omega_2\omega_2 = 0 \quad\text{..........}(74),$$

$$\omega_1\Omega_2-\Omega_2\omega_1 = \omega_2 \quad\text{..........}(75),$$

$$\omega_2\Omega_1-\Omega_1\omega_2 = \omega_1 \quad\text{..........}(76),$$

$$\omega_1\frac{d}{dy}-\frac{d}{dy}\omega_1 = \Omega_1 \quad\text{..........}(77),$$

$$\omega_2\frac{d}{dz}-\frac{d}{dz}\omega_2 = \Omega_2. \quad\text{..........}(78),$$

$$\omega_1\frac{d}{dz}-\frac{d}{dz}\omega_1 = w_1+2w_2-i \quad\text{..........}(79),$$

$$\omega_2\frac{d}{dy}-\frac{d}{dy}\omega_2 = 2w_1+w_2-i \quad\text{..........}(80),$$

$$\omega_1V_1-V_1\omega_1 = 0 \quad\text{..........}(81),$$

$$\omega_2V_2-V_2\omega_2 = 0 \quad\text{..........}(82),$$

$$\omega_1V_2-V_2\omega_1 = \Sigma\left\{\Sigma\left(\overline{r+s-1}\,x_{r,s}\,x_{m-r,n-s}\right)\frac{d}{dx_{mn}}\right\}$$

$$= \omega_2V_1-V_1\omega_2 \quad\text{..........}(83),$$

the function operated on being one of second and higher derivatives of x with regard to the independent variables y and z, and in the case of (79) and (80), one which is homogeneous and doubly isobaric. In case of all but the four (77) to (80), the function may involve also the variables.

A close analogy between the properties of ω_1, ω_2 and V_1, V_2 will be observed on comparing many of the above identities with corresponding ones obtained in my previous papers in Vols. XVII., XVIII., XIX. of the *Proceedings*. It may be well to rewrite these here, especially as where they were obtained it was convenient to use x

and y as the independent variables instead of y and z as at present, and as it is now desirable, in order to use the two sets of results in connection, to have them, too, in the present notation before us. They are

$$\Omega_1 V_1 - V_1 \Omega_1 = 0 \quad\text{.........(84)},$$

$$\Omega_2 V_2 - V_2 \Omega_2 = 0 \quad\text{.........(85)},$$

$$\Omega_1 V_2 - V_2 \Omega_1 = V_1 \quad\text{.........(86)},$$

$$\Omega_2 V_1 - V_1 \Omega_2 = V_2 \quad\text{.........(87)},$$

from Vol. XIX., p. 9, and

$$V_1 \frac{d}{dy} - \frac{d}{dy} V_1 = 2x_{20}(i+w_1) + x_{11}\Omega_1 \quad\text{.........(88)},$$

$$V_2 \frac{d}{dz} - \frac{d}{dz} V_2 = x_{11}\Omega_2 + 2x_{02}(i+w_2) \quad\text{.........(89)},$$

$$V_1 \frac{d}{dz} - \frac{d}{dz} V_1 = x_{11}(i+w_1) + 2x_{02}\Omega_1 \quad\text{.........(90)},$$

$$V_2 \frac{d}{dy} - \frac{d}{dy} V_2 = 2x_{20}\Omega_2 + x_{11}(i+w_2) \quad\text{.........(91)},$$

$$\Omega_1 \frac{d}{dy} - \frac{d}{dy}\Omega_1 = 0 \quad\text{.........(92)},$$

$$\Omega_2 \frac{d}{dz} - \frac{d}{dz}\Omega_2 = 0 \quad\text{.........(93)},$$

$$\Omega_1 \frac{d}{dz} - \frac{d}{dz}\Omega_1 = \frac{d}{dy} \quad\text{.........(94)},$$

$$\Omega_2 \frac{d}{dy} - \frac{d}{dy}\Omega_2 = \frac{d}{dz} \quad\text{.........(95)},$$

from Vol. XIX., p. 380. To these add

$$\Omega_1\Omega_2 - \Omega_2\Omega_1 = w_1 - w_2 \quad\text{.........(96)},$$

and

$$V_1 V_2 - V_2 V_1 = 0 \quad\text{.........(97)},$$

of which last, as it is far from obvious, a proof is appended. The symbolical notation of my paper in the *Proceedings*, Vol. XVIII., pp. 142, &c., is used, so that $\eta^r\zeta^s$ in an expanded result means $\frac{d}{dx_{rs}}$.

Let c_{mn} denote the coefficient of $\eta^m\zeta^n$ in

$$(\xi - x_{10}\eta - x_{01}\zeta)^2 = \{\Sigma(x_{mn}\xi^m\eta^n)\}^2,$$

which, for shortness, call μ^2. Then

$$V_1 = \tfrac{1}{2}\frac{d}{d\eta}(\mu^2) \quad \text{and} \quad V_2 = \tfrac{1}{2}\frac{d}{d\zeta}(\mu^2),$$

therefore $4V_1V_2 = \left\{\Sigma\left[(r+1)\,c_{r+1,s}\,\eta^r\zeta^s\right]\right\}\left\{\Sigma\left[(n+1)\,c_{m,n+1}\,\eta^m\zeta^n\right]\right\}$.

Therefore $\quad$ co. $\eta^m\zeta^n$ in $4(V_1V_2 - V_2V_1)$

$$= \left\{(n+1)\,\Sigma\left[(r+1)\,c_{r+1,s}\,x_{m-r,n+1-s}\right]\right.$$
$$\left. -(m+1)\,\Sigma\left[(s+1)\,c_{r,s+1}\,x_{m+1-r,n-s}\right]\right\}$$

$$= (n+1)\text{ co. }\eta^m\zeta^{n+1}\text{ in }\frac{d}{d\eta}(\mu^2)\,\mu - (m+1)\text{ co. }\eta^{m+1}\zeta^n\text{ in }\frac{d}{d\eta}(\mu^2)\,\mu$$

$$= \text{co. }\eta^m\zeta^n\text{ in }\left\{\frac{d}{d\zeta}\left(\mu^2\frac{d}{d\eta}\mu\right) - \frac{d}{d\eta}\left(\mu^2\frac{d}{d\zeta}\mu\right)\right\}$$

$$= \text{co. }\eta^m\zeta^n\text{ in }\left(2\mu\frac{d\mu}{d\eta}\frac{d\mu}{d\zeta} + \mu^2\frac{d^2\mu}{d\eta\,d\zeta} - 2\mu\frac{d\mu}{d\zeta}\frac{d\mu}{d\eta} - \mu^2\frac{d^2\mu}{d\eta\,d\zeta}\right)$$

$= 0$, for all values of m and n.

Hence (97) follows. We have thus evaluated all the alternants of pairs of $\frac{d}{dx}$, $\frac{d}{dy}$, ω_1, ω_2, Ω_1, Ω_2, V_1, V_2, the only one not written down above being the elementary identity

$$\frac{d}{dy}\cdot\frac{d}{dz} - \frac{d}{dz}\cdot\frac{d}{dy} = 0.$$

15. A few only of the many conclusions which are involved in the alternant identities before us will be added to those detailed in my former papers.

(i.) *If a function P is annihilated by ω_1, so are Ω_1P, V_1P, and ω_2P.*

(ii.) *If a function P is annihilated by ω_2, so are Ω_2P, V_2P, and ω_1P.*

(iii.) *If a function P is annihilated by both ω_1 and ω_2, so are both Ω_1P and Ω_2P.*

(iv.) *If a function P is annihilated by V_1, so are Ω_1P, ω_1P, and V_2P.*

(v.) *If a function P is annihilated by V_2, so are Ω_2P, ω_2P, and V_1P.*

(vi.) *If a function P is annihilated by both V_1 and V_2, so are both Ω_1P and Ω_2P.*

This last is stated as the companion of (iii.), but I have given and applied it in an earlier paper.

(vii.) *If a function P is annihilated by Ω_1, so are $V_1 P$, $\frac{d}{dy}P$, and $\omega_1 P$.*

(viii.) *If a function P is annihilated by Ω_2, so are $V_2 P$, $\frac{d}{dz}P$, and $\omega_2 P$.*

(ix.) *If ω_1 and Ω_2 annihilate a function, so does ω_2.*

(x.) *If ω_2 and Ω_1 annihilate a function, so does ω_1.*

(xi.) *If ω_1, V_1, Ω_1, Ω_2 annihilate a function, so do ω_2 and V_2.*

The last fact tells us, as was stated in Article 12, that annihilation by ω_1, V_1, Ω_1, Ω_2 is enough to certify a projective cyclicant.

16. *Generation of seminvariants from other seminvariants and invariants.*

From (vii.) above we learn that, operating on any seminvariant or invariant of the forms

$$\left.\begin{array}{l}(z_{20}, z_{11}, z_{02}\between (u, v)^2 \\ (z_{30}, z_{21}, z_{12}, z_{03}\between (u, v)^3 \\ \quad \&c. \qquad \&c.\end{array}\right\} \qquad (98),$$

ω_1 generates another seminvariant of the system. That the operators V_1 and $\frac{d}{dy}$ have the same property, I have mentioned in earlier papers.

It is to be noticed that the three generators produce from any given seminvariant other seminvariants of quite different types. Thus,

(α) V_1 generates from a given seminvariant another of higher degree, the same weight (second partial weight), and eventually (after a succession of operations) one of a diminished number of the forms.

(β) $\frac{d}{dy}$, y being the first of the two independent variables, generates one of the same degree, of the same weight (second partial weight), and of an increased number of forms.

(γ) ω_1 generates one of the same degree, *diminished* weight (second partial weight), and eventually (upon repeated application) one of a diminished number of forms.

I hope on some future occasion to deal with these and other simple generators of seminvariants, *e.g.*, separate parts of the above generators, at greater length.

17. *Generation of pure persistents for the transformation* (11) *from others.*

If ω_1 and Ω_1 annihilate a pure function P, then P is a pure persistent. Now, by (77) and (92), ω_1 and Ω_1 under these circumstances annihilate also $\frac{dP}{dy}$; and, by (72) and (76), they also annihilate $\omega_2 P$. Consequently $\frac{d}{dy}$ and ω_2, *i.e.*, ω_2 and

$$\frac{d}{dy} = \Sigma \left\{ (m+1)\, x_{m+1,n} \frac{d}{dx_{mn}} \right\} \quad \text{.................}(99),$$

are generators of pure persistents for the transformation (11) from other such persistents.

18. *Generators of projective semicyclicants from other projective semicyclicants or cyclicants.*

If ω_1, Ω_1, and V_1 annihilate a pure function P, it is a projective semicyclicant; or a projective cyclicant if it is also annihilated by Ω_2, *i.e.*, if its two partial weights are equal.

Now (77) and (92) tell us, as in the last article, that ω_1 and Ω_1 must also annihilate $\frac{dP}{dy}$; and (88) tells us that, if only $i+w_1 = 0$, V_1 will also annihilate $\frac{dP}{dy}$. Consequently, the operator $\frac{d}{dy}$, applied to a projective semicyclicant or cyclicant the sum of whose degree and first partial weight vanishes, generates another projective semicyclicant. That from any semicyclicant it generates a semicyclicant I have previously shown (*Proceedings*, Vol. XIX., p. 382).

Now, $P(i, w_1, w_2)$ being any projective cyclicant or semicyclicant of type i, w_1, w_2, since x_{20} is another, it follows that

$$\frac{P(i, w_1, w_2)}{x_{20}^{\frac{1}{2}(i+w_1)}}$$

is one whose degree and first partial weight have a vanishing sum. It follows that

$$\frac{d}{dy}\, \frac{P(i, w_1, w_2)}{x_{20}^{\frac{1}{2}(i+w_1)}} \quad \text{......................}(100)$$

is another projective semicyclicant, which when multiplied by $x_{20}^{\frac{1}{2}(i+w_1)+1}$ produces one that is integral, its type being $(i+1, w_1+3, w_2)$.

But

$$\frac{d}{dy}\left\{\frac{P(i,\, w_1,\, w_2)}{x_{20}^{\frac{1}{2}(i+w_1)}}\right\}=\frac{1}{x_{20}^{\frac{1}{2}(i+w_1)+1}}\left\{x_{20}\frac{d}{dy}-\tfrac{1}{2}(i+w_1)\,3x_{30}\right\}P(i,\, w_1,\, w_2)$$

$$=\frac{1}{x_{20}^{\frac{1}{2}(i+w_1)+1}}\left\{x_{20}\Sigma\left[(m+1)\,x_{m+1,n}\frac{d}{dx_{mn}}\right]\right.$$

$$\left.-x_{30}\Sigma\left[(m+1)\,x_{mn}\frac{d}{dx_{mn}}\right]\right\}P(i,\, w_1,\, w_2)$$

$$=\frac{1}{x_{20}^{\frac{1}{2}(i+w_1)+1}}\Sigma\left\{(m+1)(x_{20}x_{m+1,n}-x_{30}x_{mn})\frac{d}{dx_{mn}}\right\}P(i,\, w_1,\, w_2).$$

Thus the quadro-linear operator

$$G=\Sigma\left\{(m+1)(x_{20}x_{m+1,n}-x_{30}x_{mn})\frac{d}{dx_{mn}}\right\}\ \ldots\ldots\ldots\ldots(101)$$

is a generator of *projective* semicyclicants from other *projective* semicyclicants and cyclicants. That it generates semicyclicants from semicyclicants, is practically shown in the passage above cited of my paper in Vol. XIX.

Two classes of projective semicyclicants have simpler generators from other semicyclicants of the same classes. These follow.

19. *Generator of projective semicyclicants having the additional property of being annihilated by V_2 from other semicyclicants of the same kind.*

If $\omega_1 P=0$, $\Omega_1 P=0$, $V_1 P=0$, $V_2 P=0$, it follows from (76) that $\Omega_1 . \omega_2 P=0$, from (72) that $\omega_1 . \omega_2 P=0$, from (83) that $V_1 . \omega_2 P=0$, and from (82) that $V_2 . \omega_2 P=0$. Thus ω_2 is a generator of semicyclicants of the kind described in the heading from other semicyclicants of the same kind.

If P be of type $(i,\, w_1,\, w_2)$, $\omega_2 P$ is of type $(i,\, w_1-1,\, w_2)$. Thus, in particular, if w_1 exceeds w_2 by unity, we may expect a projective cyclicant to be generated.

20. *Generator of principiant semicyclicants from principiant semicyclicants or from cyclicants.*

Principiant semicyclicants have been defined as those projective semicyclicants which ω_2, as well as ω_1, Ω_1, V_1, annihilates.

V_2 is a generator of such from such or from projective cyclicants; for, as in the last article, we see, by means of (82), (86), (83), and (97), that if ω_1, ω_2, Ω_1, V_1 all annihilate a pure function P, they all annihilate $V_2 P$.

If P is of type $(i,\, w_1,\, w_2)$, $V_2 P$ is of type $(i+1,\, w_1,\, w_2+1)$.

21. *Projective and principiant cocyclicants.*

A cocyclicant is, it will be remembered, a covariant of the cyclicogenitive forms

$$\left.\begin{aligned} E_2 &= (z_{20}, z_{11}, z_{02} \between (-z_{01}, z_{10})^2 \\ E_3 &= (z_{30}, z_{21}, z_{12}, z_{03} \between (-z_{01}, z_{10})^3 \\ E_4 &= (z_{40}, z_{31}, z_{22}, z_{13}, z_{04} \between (-z_{01}, z_{10})^3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (102),$$

&c. &c.

whose leading coefficient is a semicyclicant; and it has been shown (Vol. XIX., pp. 379 and 20) that, $(S_0, S_1, \ldots S_m)(-z_{01}, z_{10})^m$ being a cocyclicant,

$$\frac{S_0(x, yz)}{x_{01}^{i+w_1}} = (-1)^m \frac{S_m(y, zx)}{y_{10}^{i+w_1}} = (-1)^{i+w_2}(S_0, S_1, \ldots S_m)(-z_{01}, z_{10})^m$$

$$\ldots\ldots\ldots\ldots\ldots (103),$$

$S_0(x, yz)$ being the corresponding semicyclicant (in x dependent).

A *projective* cocyclicant is now defined as one whose leading coefficient is a *projective* semicyclicant, and in particular a *principiant* cocyclicant as one whose leading coefficient is a *principiant* semicyclicant.

In proving that every projective, or in particular principiant, semicyclicant in x persists, but for a factor involving first derivatives and variables only, after the special homographic transformation (71), we have, in virtue of (103), proved the same property of persistence to belong to all projective, and in particular principiant, cocyclicants in z.

22. *Some instances of projective cyclicants, semicyclicants, &c.*

It is *à priori* clear, and has been noticed by Halphen, that in the criteria of developable and ruled surfaces we must have two projective or principiant pure cyclicants, persisting for all homographic transformations.

The two are $\quad x_{20}x_{02} - \frac{1}{4}x_{11}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (104),$

and

$$\begin{vmatrix} x_{30}, & x_{21}, & x_{12}, & x_{03} & \\ & x_{30}, & x_{21}, & x_{12}, & x_{03} \\ x_{20}, & x_{11}, & x_{02} & & \\ & x_{20}, & x_{11}, & x_{02} & \\ & & x_{20}, & x_{11}, & x_{02} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (105).$$

Both have been seen in earlier papers to be annihilated by Ω_1, Ω_2, V_1, V_2. The further conditions, that ω_1 and (therefore also) ω_2 annihilate them, are easily verified. ω_1, for instance, produces the fourth row of (105) from the first, and the fifth from the second.

The operator G, (101), annihilates (104). By repeated operation on (105), it will however produce a succession of projective semicyclicants.

23. The simplest of all semicyclicants is x_{20}. This is annihilated by ω_1, ω_2, and V_2. It is, then, a principiant semicyclicant having the additional property of being annihilated by V_2. Consequently, the quadratic cyclicogenitive form

$$(z_{20}, z_{11}, z_{02}\between(-z_{01}, z_{10})^2 = -\frac{x_{20}}{x_{01}^2} \quad \text{.................}(106)$$

is a principiant cocyclicant with the additional V_2 property.

The other cyclicogenitive forms are not cocyclicants at all.

The generators G, ω_2, V_2 produce nothing from the principiant semicyclicant x_{20}.

24. Of the other simple semicyclicants and cocyclicants given in my last paper (Vol. XIX., pp. 392—398), most are projective and indeed principiant. This is not the case with $3x_{30}x_{11}-2x_{20}x_{21}$, which ω_1 does not annihilate. It is easily verified, however, that the semicyclicant of Vol. XIX., p. 395 (59), viz., $G(3x_{30}x_{11}-2x_{20}x_{21})$, or

$$Q = \begin{vmatrix} 2x_{20}, & x_{11} & \\ 3x_{30}, & x_{21}, & x_{20} \\ 4x_{40}, & x_{31}, & 2x_{30} \end{vmatrix} \quad \text{.......................}(107),$$

is annihilated by both ω_1 and ω_2. This, then, is a principiant semicyclicant, and the corresponding covariant in z of the cyclicogenitive forms, viz.,

$$\left(Q,\ \frac{1}{6}\Omega_2 Q,\ \frac{1}{6.5}\Omega_2^2 Q,\ \dots\ \frac{1}{6\,!}\Omega_2^6 Q\right)(-z_{01}, z_{10})^6 \quad \text{......}(108),$$

is a principiant cocyclicant.

G, G^2, G^3, ..., operating on (107), produce other projective (not necessarily principiant) semicyclicants of higher degree and first partial weight. Again, V_2, V_2^2, V_2^3, ... operating upon it produce a limited number of other principiant semicyclicants.

The family of surfaces whose criterion is Q, or the left-hand member of (108), has been alluded to in Vol. XIX., pp. 395, 396. It is now

seen that the family includes more than has been there stated, viz., in fact, surfaces which cut planes parallel to $z = 0$ in curves in perspective, and not merely in homothetic curves.

25. The semicyclicant $\begin{vmatrix} x_{30}, & x_{20} & \\ x_{21}, & x_{11}, & x_{20} \\ x_{12}, & x_{02}, & x_{11} \end{vmatrix}$ ....(109),

of Vol. XIX., p. 397, § 18, or p. 13 (10), is annihilated by ω_1 and ω_2 as well as Ω_1, V_1, and V_2. It is then a principiant semicyclicant having the extra property of being annihilated by V_2, and its corresponding cocyclicant in z,

$$\begin{vmatrix} z_{30}, & z_{20}, & & z_{10}^3 \\ z_{21}, & z_{11}, & z_{20}, & 3z_{10}^2 z_{01} \\ z_{12}, & z_{02}, & z_{11}, & 3z_{10}z_{01}^2 \\ z_{03}, & & z_{02}, & z_{01}^3 \end{vmatrix} \qquad \text{......................(110)},$$

is a principiant cocyclicant whose leading coefficient is annihilated by V_2.

Operation on (109) with G, G^2, G^3, ... produces other projective, not shown to be principiant, semicyclicants. The other generators ω_2, V_2 are here annihilators.

26. Of the semicyclicants and cocyclicants discussed in Vol. XIX., pp. 398—405, which are of second partial weight zero, and one of which is obtained from every Sylvesterian pure reciprocant, all are projective, being annihilated by ω_1, and not merely those which are obtained from projective or principiant reciprocants. These last have the property of being also annihilated by ω_2. It is this fact which has led me to distinguish between two classes of projective semicyclicants, and call the more restricted class which ω_2 annihilates by Professor Sylvester's second name *principiant*.

The present conclusion may be stated concisely by saying that homogeneous and isobaric functions of the cyclicogenitive forms E_2, E_3, E_4, ..., which have the annihilator

$$4\frac{E_2^2}{2}\frac{d}{dE_3}+5E_2E_3\frac{d}{dE_4}+6\left(E_2E_4+\tfrac{1}{2}E_3^2\right)\frac{d}{dE_5}+7\left(E_2E_5+E_3E_4\right)\frac{d}{dE_6}+\dots$$

..............(107),

are *projective* cocyclicants; and that *principiant* cocyclicants which

are functions of E_2, E_3, E_4, ... have also the annihilator

$$E_2\frac{d}{dE_3}+2E_3\frac{d}{dE_4}+3E_4\frac{d}{dE_5}+\dots \quad \dots\dots\dots\dots\dots(108).$$

27. *Formation of projective or principiant cyclicants.*

It will now be proved that—*Every invariant of a principiant cocyclicant whose semicyclicant source is also annihilated by V_2, considered as a quantic in $-z_{01}$, z_{10}, is a projective cyclicant.*

If S be the semicyclicant source of such a cocyclicant, it is annihilated by ω_1, ω_2, Ω_1, V_1, V_2. $\Omega_2 S$ has then the annihilators ω_1, ω_2, V_1, V_2, by (75), (74), (87), (85). $\Omega_2^2 S$ has the same annihilators by the same identities, and so on. Thus all the coefficients of the cocyclicant have the annihilators ω_1, ω_2, V_1, V_2, and consequently any function of them has the same. Now, an invariant of the cocyclicant, being an invariant of the cyclicogenitive forms of which that cocyclicant is a covariant, is annihilated by Ω_1 and Ω_2. The invariant has then all the annihilators ω_1, ω_2, V_1, V_2, Ω_1, Ω_2, and is consequently a projective cyclicant—a persistent or differential invariant for the general homographic transformation.

It may happen that a seminvariant of the cocyclicant occurs which is annihilated by Ω_2, through breaking up into factors, one of which is a function of second partial weight zero, and the other an invariant of the cyclicogenitive forms. Such seminvariants, with the first factor rejected, also give projective cyclicants.

We have examples of cocyclicants which yield principiant cyclicants in this manner in the quadratic cyclicogenitive form and in (110). For their discussion, see Vol. XIX., p. 16.

Other principiant cyclicants with the property in question are to be expected to be produced from any homogeneous and doubly isobaric forms of which ω_1, ω_2, V_1, V_2 are annihilators, as in Vol. XIX., p. 14.

I am not sure that this method will be very productive.

28. The following two propositions afford other methods for the production of projective cyclicants.

(*a*) *If S be a projective semicyclicant of type i, w_1, w_2, which is annihilated by V_2 as well as by ω_1, Ω_1, V_1, then $\omega_2^{w_1-w_2}S$ is a projective cyclicant.*

(*b*) *If S' be a principiant semicyclicant, i.e., one annihilated by ω_2 as well as by ω_1, Ω_1, V_1, of type i, w_1, w_2, then $V_2^{w_1-w_2}S'$ is a projective cyclicant.*

By Art. 19, if ω_1, Ω_1, V_1, V_2 annihilate S, they also annihilate $\omega_2 S$, $\omega_2^2 S$, $\omega_2^3 S$, &c., &c. Now, since the operation ω_2 diminishes w_1, by unity and leaves w_2 unaltered, the two partial weights of $\omega_2^{w_1-w_2} S$ are equal. Consequently, by (96), Ω_2 annihilates $\omega_2^{w_1-w_2} S$; and hence also, by (75), so does ω_2. $\omega_2^{w_1-w_2} S$ is then a projective cyclicant.

Again, if ω_1, ω_2, Ω_1, V_1 annihilate S', they annihilate $V_2 S'$, by Art. 20. In like manner they annihilate $V_2^2 S'$, $V_2^3 S'$, &c., &c. Now, V_2 increases w_2 by unity, and does not alter w_1. Thus the two partial weights of $V_2^{w_1-w_2} S'$ are equal; so that, by (96), Ω_2 annihilates $V_2^{w_1-w_2} S'$, and consequently so does V_2, by (87). Thus $V_2^{w_1-w_2} S'$ is a projective cyclicant.

The only possibility which may interfere with the success of these methods is that $\omega_2^{w_1-w_2}$ and $V_2^{w_1-w_2}$ be annihilators under the circumstances considered. I cannot see, however, that the complete system of alternant identities (72)—(96) gives us any reason for fearing this to be generally the case.*

29. One more method for the systematic calculation of projective semicyclicants and cyclicants will be exhibited. It has been shown (Vol. XIX., pp. 22, 23) how to obtain all the linearly independent pure cyclicants of a given type $\left(i, \frac{w}{2}, \frac{w}{2}\right)$; and the method has been extended (Vol. XIX., p. 379) to the obtaining of all the linearly independent semicyclicants of a given type (i, w_1, w_2). It has been proved, also, that a superior limit to the, and it may be the exact, number of these pure cyclicants or semicyclicants is the excess of the number of seminvariants of type (i, w_1, w_2) of the cyclicogenitive forms over the number of type $(i+1, w_1+1, w_2)$.

Now, let S_1, S_2, S_3, ... be a complete system of semicyclicants, or pure cyclicants of the type (i, w_1, w_2) thus determined, and let S_1', S_2', S_3', ... be a complete system of type (i, w_1, w_2-1). By (73) and (84), the operation of ω_1 on a semicyclicant generates another semicyclicant. Thus we must have

$$\omega_1 (a_1 S_1 + a_2 S_2 + a_3 S_3 + \dots) \dots\dots\dots\dots\dots\dots (109),$$

* Should it be the case, we are none the less helped in our search for projective cyclicants. Suppose, for instance, that $\omega_2^m S = 0$ or $V_2^m S' = 0$, m being not greater than $w_1 - w_2$. Then $\omega_2^{m-1} S$ or $V_2^{m-1} S'$, as the case may be, is annihilated by ω_1, ω_2, V_1, V_2, Ω_1, in virtue of Art. 18 or 19. Consequently $\omega_2^{m-1} S$, or $V_2^{m-1} S'$, as the case may be, is the leading coefficient of such a cocyclicant as has, by Art. 26, the property that any one of its invariants is a projective cyclicant.

equal to such an expression as

$$(la_1+ma_2+na_3+\ldots)\,S_1'+(l'a_1+m'a_2+n'a_3+\ldots)\,S_2'$$
$$+(l''a_1+m''a_2+n''a_3+\ldots)\,S_3'+\ldots.$$

For ω_1 to be an annihilator of the sum on which it operates in (109), we must have simultaneously

$$la_1+ma_2+na_3+\ldots=0,$$
$$l'a_1+m'a_2+n'a_3+\ldots=0,$$
$$l''a_1+m''a_2+n''a_3+\ldots=0, \text{ \&c. \&c.},$$

a number of equations for the determination of $a_1, a_2, a_3, \ldots$ less than the number of those coefficients by the excess of the number of $S_1, S_2, \ldots$ over the number of $S_1', S_2', \ldots$.

By this means are found all the functions of the type (i, w_1, w_2) which ω_1 as well as Ω_1 and V_1 annihilate, *i.e.*, all the projective semicyclicants of the type. If $w_1=w_2$, then, by (96), Ω_2 is also an annihilator, and the functions are projective cyclicants.

If $N(i, w_1, w_2)$ denote the number of linearly independent seminvariants of type (i, w_1, w_2) of the cyclicogenitive forms, it is thus seen that the number of linearly independent projective semicyclicants of the type, or cyclicants if $w_1=w_2$, is likely, though not certain, to prove to be

$$N(i, w_1, w_2)-N(i+1, w_1+1, w_2)-N(i, w_1, w_2-1)$$
$$+N(i+1, w_1+1, w_2-1) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(110).$$

30. The resemblance of the operators $\omega_1, \omega_2, \Omega_1, \Omega_2$ to Mr. Forsyth's $\Delta_1, \Delta_2, \Delta_4, \Delta_5$ which, annihilate functions of the derivatives that are invariantic for homographic transformations of the independent variables only, is striking, but must not mislead. The distinction between ω_1, ω_1 and Δ_4, Δ_5, though at first sight slight, is essential. It is a remarkable conclusion from one of Mr. Forsyth's theorems that none of his invariants can be pure cyclicants, for all of them involve first derivatives. The important memoir here referred to is published in the *Phil. Trans.*, Vol. CLXXX., pp. 71—118.

On Secondary Invariants. *By* L. J. ROGERS.

[*Read Feb.* 14*th*, 1889.]

CONTENTS.

In a paper I had the honour to read before this Society two years ago, I showed the existence of a certain class of reciprocants to which I gave the name "homographic," inasmuch as they were unaltered except for numerical factors and powers of $\frac{dy}{dx}$ when the variables x and y were affected by homographic change, that is, when x became $\frac{ax+b}{x+c}$, and y became $\frac{ay+\beta}{y+\gamma}$.

In other words, the equating to zero of such reciprocants led to complete primitive equations of the form

$$\frac{\alpha y+\beta}{y+\gamma}=f\left(\frac{ax+b}{x+c}\right),$$

where f was some self-reciprocating function.

The peculiar outward properties of such reciprocants were: (1) homogeneity, (2) isobarism, (3) annihilation under two differential operators, none of which properties were tests for reciprocance. In fact, all the properties held good irrespective of any reciprocant ideas, provided the complete primitive was similar to that written above, where f might be any function whatever. In this case the homogeneous, isobaric, and doubly annihilable functions of

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ are called doubly homographic invariants, and have been investigated by Mr. Forsyth in the *Messenger of Mathematics*, for February, March, and April, 1888. There, by the simple substitution of $a_0, a_1, a_2 \ldots$ for $\frac{dy}{dx}, \frac{1}{\underline{|2}} \frac{d^2y}{dx^2}, \frac{1}{\underline{|3}} \frac{d^3y}{dx^3} \ldots$, such homographic invariants become homogeneous and isobaric functions of the a's annihilable by

$$a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \ldots,$$

which I propose to write O_1, and by

$$a_0^2 \frac{d}{da_1} + 2a_0 a_1 \frac{d}{da_2} + (2a_0 a_2 + a^2) \frac{d}{da_3} + \ldots,$$

which I shall call O_2.

The symbol δ_n will also be used for $\frac{d}{da_n}$.

The coefficients in O_2 are easily seen to be the coefficients of ascending powers of x in $(a_0 + a_1 x + a_2 x^2 + \ldots)^2$, and for brevity will be written $A_0, A_1, A_2 \ldots$ Thus O_2 will be written

$$A_0 \delta_1 + A_1 \delta_2 + A_2 \delta_3 + \ldots$$

Since doubly homographic invariants are annihilable by O_1, their form is identical with that of algebraic semi-invariants, or, as Prof. Sylvester calls them, ordinary invariants.

I propose to call all homogeneous and isobaric functions of the a's *primary* invariants if annihilated by O_1, *secondary* if annihilated by O_2, and *double* invariants if annihilated by both O_1 and O_2.

If invariants are also annihilated by an operator of the form

$$na_1 \delta_0 + (n-1)\, a_2 \delta_1 + (n-2)\, a_3 \delta_2 + \ldots,$$

Prof. Sylvester calls it *satisfied* for the order n.

The existence of satisfied double invariants, *i.e.* such as are annihilated by O_1, O_2, and Ω_n, has been proved in Mr. Forsyth's paper in the *Messenger*, in which he has shown that

$$a_0 a_2 - a_1^2, \quad a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^3,$$

and in fact all the catalecticants of even-ordered quantics from our present point of view, give, on being equated to zero, the complete primitives

$$(ax+b)\, y = x + c, \quad (ax^2 + bx + c)\, y = x^2 + dx + e, \ \&c.$$

These are known to have the forms of satisfied invariants, while the double homographic property of their primitives proves their annihilation by O_2.

1. It will be necessary first to consider the result of operating on the coefficients A_0, A_1, ..., with the two operations

$$a_1\delta_0+2a_2\delta_1+3a_3\delta_2+\dots,$$

and

$$a_1\delta_0+a_2\delta_1+a_3\delta_2+\dots.$$

Consider the identity

$$(a_0+a_1x+a_2x^2+\dots)^2 = A_0+A_1x+\dots.$$

Now, a change of

$$F(a_0, a_1, a_2 \dots) \text{ into } F(a_0, a_1 \dots)+\lambda\,(a_1\delta_0+2a_2\delta_1+\dots)\,F$$

denotes a change of

$$\begin{aligned} a_0 &\text{ into } a_0+\lambda a_1, \\ a_1 &\quad ,, \quad a_1+2\lambda a_2, \\ a_2 &\quad ,, \quad a_2+3\lambda a_3, \end{aligned}$$

where λ is so small that we neglect λ^2, λ^3, &c., and consequently

$$a_0+a_1x+\dots \text{ becomes } (a_0+\lambda a_1)+(a_1+2\lambda a_2)\,x+\dots,$$

i.e.,

$$a_0+a_1\,(x+\lambda)+a_2\,(x^2+2\lambda x)+\dots,$$

which is equivalent to saying that x becomes $x+\lambda$. But this will make

$$A_0+A_1x+\dots \text{ become } A_0+A_1\,(x+\lambda)+\dots,$$

that is,

$$(A_0+\lambda A_1)+(A_1+2\lambda A_2)\,x+\dots.$$

Consequently the operation of $a_1\delta_0+2a_2\delta_1+\dots$

$$\begin{aligned} \text{on } A_0 &\text{ gives } A_1, \\ A_1 &\quad ,, \quad 2A_2, \\ A_2 &\quad ,, \quad 3A_3, \text{ \&c.} \end{aligned} \qquad\qquad (1).$$

Similarly, for the second operation, we change a_0 into $a_0+\lambda a_1$, a_1 into $a_1+\lambda a_2$, &c., so that $A_0+A_1x+\dots$ becomes

$$\begin{aligned} &\left\{(a_0+\lambda a_1)+(a_1+\lambda a_2)\,x+\dots\right\}^2 \\ &= \left\{(a_0+a_1x+a_2x^2+\dots)\left(1+\frac{\lambda}{x}\right)-\frac{a_0\lambda}{x}\right\}^2 \\ &= (A_0+A_1x+\dots\dots\dots)\left(1+\frac{2\lambda}{x}\right)-\frac{2a_0}{x}\,(a_0+a_1x+a_2x^2+\dots) \\ &= (A_0+2\lambda A_1-2a_0a_1\lambda)+(A_1+2\lambda A_2-2a_0a_2\lambda)\,x+\dots. \end{aligned}$$

so that the operation of $a_1\delta_0+a_2\delta_1+a_3\delta_2+\dots$

$$\begin{aligned} \text{on } A_0 &\text{ gives } 2A_1-2a_0a_1, \\ A_1 &\quad ,, \quad 2A_2-2a_0a_2, \\ A_2 &\quad ,, \quad 2A_3-2a_0a_3, \ \&c. \quad \dots\dots\dots\dots(2). \end{aligned}$$

The effect of O_1 operating on A_0, A_1 may also be found thus. We have identically

$$\left(\frac{a_0}{x}+\frac{a_1}{x^2}+\dots\right)^2 = \frac{A_0}{x^2}+\frac{A_1}{x^3}+\dots.$$

Change x into $x-\lambda$, where λ is very small. Then

$$\left\{\frac{a_0}{x}\left(1+\frac{\lambda}{x}\right)+\frac{a_1}{x^2}\left(1+\frac{2\lambda}{x}\right)+\dots\right\}^2 = \frac{A_0}{x^2}\left(1+\frac{2\lambda}{x}\right)+\dots$$

or $$\left(\frac{a_0}{x}+\frac{a_1+\lambda a_0}{x^2}+\frac{a_2+2\lambda a_1}{x^3}+\dots\right)^2 = \frac{A_0}{x^2}+\frac{A_1+2\lambda A_0}{x^3}+\dots$$

Hence the effect of $a_0\delta_1+2a_1\delta_2+\dots$

$$\begin{aligned} \text{on } A_0 &\text{ is nothing,} \\ A_1 &\text{ is } 2A_0, \\ A_2 &\text{ is } 3A_1, \ \&c. \dots\dots\dots\dots(3). \end{aligned}$$

These results are of the utmost importance in the following pages.

2. *Generators.*

In my first memoir on Reciprocants, I showed the existence of a generator for differential expressions which were not essentially altered when y became $\dfrac{ay+b}{a+c}$. This generator was

$$y_1\frac{d}{dx}-iy_2,$$

and this becomes, when the y's are changed into the a's,

$$a_0(2a_1\delta_0+3a_2\delta_1+\dots)-2a_1(a_0\delta_0+a_1\delta_1+\dots),$$

which will therefore be a generator for secondary invariants.

The fact that this is a generator is very easy to verify. We merely have to show that, if $O_2J=0$, (so that J is a secondary invariant), then also will $O_2GJ=0$, *i.e.* $(GO_2-O_2G)J=0$. Now this alternant operation is such that all differential operations of the second order

cancel, and by § 1 (1) and (2), we easily get

$$(GO_2 - O_2G)\,J = 2a_1 O_2 J,$$

which is zero by hypothesis.

Though this is in form the simplest secondary generator, yet for our present purpose it is not the most interesting. I shall, in fact, make exclusive use of the expression

$$a_0(2a_1\delta_0 + 3a_2\delta_1 + \ldots) - A_1\delta_0 - A_2\delta_1 - \ldots \qquad \ldots\ldots\ldots\ldots(1),$$

which I shall call Γ.

We have first to show that it is a secondary generator, *i.e.*, that if

$$O_2 J = 0,$$

then also $\quad O_2\Gamma J = 0, \quad$ or $\quad (O_2\Gamma - \Gamma O_2)\,J = 0.$

$$\begin{aligned}
\text{Now}\quad O_2\Gamma - \Gamma O_2 &= (A_0\delta_1 + A_1\delta_2 + \ldots)\,\{a_0(2a_1\delta_0 + \ldots) - A_1\delta_0 - \ldots\} \\
&\qquad - \{a_0(2a_1\delta_0 + \ldots) - A_1\delta_0 - \ldots\}\,(A_0\delta_1 + \ldots) \\
&= \Sigma\left[O_2\{(n+2)\,a_0a_{n+1} - A_{n+1}\} - \{a_0(2a_1\delta_0 + \ldots) - A_1\delta_0 - \ldots\}\,A_{n-1}\right]\delta_n \\
&= \Sigma\big[(n+2)\,a_0A_n - 2A_0a_n - 2A_1a_{n-1} - \ldots - 2A_na_0 \\
&\qquad - a_0(2a_1\delta_0 + 3a_2\delta_1 + \ldots)\,A_{n-1} \\
&\qquad + 2A_1a_{n-1} + 2A_2a_{n-2} + \ldots + 2A_na_0\big]\,\delta_n \\
&= \Sigma\left[(n+2)\,a_0A_n - 2A_0a_n - a_0(2a_1\delta_0 + \ldots)\,A_{n-1}\right]\delta_n,
\end{aligned}$$

which, by § 1, (1) and (2), reduces to

$$\Sigma\left[(n+2)\,a_0A_n - 2A_0a_n - na_0A_n - 2a_0A_n + 2a_0^2a_n\right]\delta_n = 0.$$

Thus Γ is shown to be a secondary generator.

Now let $\qquad a_0a_2 - a_1^2 = K_2,$

and

$$\begin{aligned}
2K_3 &= \Gamma K_2, \\
3K_4 &= \Gamma K_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2), \\
&\&c.,
\end{aligned}$$

whence we can easily find

$$\begin{aligned}
K_3 &= a_0^2a_3 - 2a_0a_1a_2 + a_1^3, \\
K_4 &= a_0^3a_4 - 2a_0^2a_1a_3 - a_0^2a_2^2 + 3a_0a_1^2a_2 - a_1^4, \\
K_5 &= a_0^4a_5 - 2a_0^3a_1a_4 - 2a_0^3a_2a_3 + 3a_0^2a_1^2a_3 + 3a_0^2a_1a_2^2 - 4a_0a_1^3a_2 + a_1^5 \ldots(3). \\
&\&c. \qquad \&c.
\end{aligned}$$

Moreover, any secondary invariant can be expressed in terms of these protomorphs. We can also see that the suffixes of the K's are identical with the weight and degree of their expanded equivalents.

In fact, we have, as in all homogeneous and isobaric functions, the two grade-equations

$$a_0\delta_0 + a_1\delta_1 + \ldots = i,$$

and

$$a_1\delta_1 + 2a_2\delta_2 + \ldots = w.$$

3. There are, however, a certain class of secondary invariants which have a very remarkable third grade-equation, viz.,

$$A_0\delta_0 + A_1\delta_1 + A_2\delta_2 + \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

which I shall call E in the following pages.

If we take the alternant of Γ and E, *i.e.*, the operation $E\Gamma - \Gamma E$, we get, by § 2, (1),

$$(A_0\delta_0 + A_1\delta_1 + \ldots)\left\{a_0(2a_1\delta_0 + \ldots) - A_1\delta_0 - \ldots\right\}$$

$$-\left\{a_0(2a_1\delta_0 + \ldots) - A_1\delta_0 - \ldots\right\}(A_0\delta_0 + \ldots)$$

$$= \Sigma\left[E\left\{(n+2)\,a_0a_{n+1} - A_{n+1}\right\} - \left\{a_0(2a_1\delta_0 + \ldots) - A_1\delta_0 - \ldots\right\}A_n\right]\delta_n,$$

just as in the last section, by § 1, (1) and (2),

$$= \Sigma\left[(n+2)(A_0a_{n+1} + a_0A_{n+1}) - 2A_0a_{n+1} - 2A_1a_n - \ldots - 2A_na_0\right.$$

$$\left.-a_0(\overline{n+1}\,A_{n+1} + 2A_{n+1} - 2a_0a_{n+1}) + 2A_1a_n + \ldots + 2A_na_0\right]\delta_n$$

$$= \Sigma\left[(n+2)\,A_0a_{n+1} - a_0A_{n+1}\right]\delta_n = a_0\Gamma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Thus the operation of $E\Gamma - \Gamma E$ is the same as that of $a_0\Gamma$. Now, by trial, we find that

$$EK_2 = 3a_0K_2,$$

and, by (2),

$$E\,.\,\Gamma K_2 - \Gamma\,.\,3a_0K_2 = a_0\Gamma K_2 = 2a_0K_3,$$

therefore

$$2EK_3 = 8a_0K_3,$$

i.e.,

$$EK_3 = 4a_0K_3.$$

Similarly, by (2),

$$E\Gamma K_3 - \Gamma\,.\,4a_0K_3 = a_0\Gamma K_3 = 3a_0K_3,$$

whence

$$EK_4 = 5a_0K_4.$$

By this process, we can easily induce that

$$EK_n = (n+1)\,a_0K_n \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This is a very remarkable equation, as it shows that some, but not all, secondary invariants have a third grade-equation, and that we have an easy test for distinguishing such. In fact they will consist, and consist only, of those invariants which can be expressed as homogeneous functions of the K's. We have already seen that K-functions must be isobaric, though they need not be homogeneous. Thus $K_4+K_2^2$ is the secondary invariant $a_0(a_0^2a_4-2a_0a_1a_3+a_1^2a_2)$, but E is not a grade-equation for the latter, since

$$E(K_4+K_2^2) = a_0(5K_4+6K_2^2).$$

On the other hand $K_4K_2-K_3^2$, which

$$= a_0^3(a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3),$$

has E for a grade-equation, since

$$E(K_4K_2-K_3^2) = 8a_0(K_4K_2-K_3^2).$$

In fact, E gives the degree + the weight of the K-function. Such secondary invariants I propose to call *ultra-homogeneous.*

4. There is also another set of K-functions which overlap this last set, and are characterised by the property that the operator

$$a_1^2\delta_1+2a_1a_2\delta_2+(2a_1a_3+a_2^2)\delta_3+\dots \qquad \dots\dots\dots\dots\dots(1),$$

which we shall call F, reproduces the K-function multiplied by a_1 and a numerical constant.

It is easily seen that F is equivalent to

$$A_1\delta_0+A_2\delta_1+\dots-2a_0(a_1\delta_0+a_2\delta_1+\dots),$$

i.e., $$= a_0(a_2\delta_1+2a_3\delta_2+3a_4\delta_3+\dots)-\Gamma, \text{ by § 2 (1).}$$

Alternating F and Γ, we get

$$\Gamma F-F\Gamma = a_0\left\{\Gamma(a_2\delta_1+2a_3\delta_2+\dots)-(a_2\delta_1+2a_3\delta_2+\dots)\Gamma\right\}$$

$$= \Sigma a_0\big[n(n+3)a_0a_{n+2}-nA_{n+2}-(n+2)(n+1)a_0a_{n+2}+(n+2)A_{n+2}$$
$$-2A_{n+2}+2a_0a_{n+2}\big]\delta_n$$

$$= 0 \text{ identically} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2).$$

Now $$FK_2 = (a_1^2\delta_1+2a_1a_2\delta_2)K_2 = a_1wK_2 = 2a_1K_2,$$

and, by (2), $$F\Gamma K_2 = \Gamma FK_2 = \Gamma\,.\,2a_1K_2.$$

Now $$\Gamma K_2 = 2K_3,$$

and $\qquad \Gamma a_1 = a_0 a_2 - a_1^2 = K_2$, see § 2 (2),

therefore $\qquad 2FK_3 = 4a_1 K_3 + 2K_2^2$,

therefore $\qquad FK_3 = 2a_1 K_3 + K_2^2$.

Again, by (2), $\qquad F\Gamma K_3 = \Gamma FK_3 = \Gamma(2a_1 K_3 + K_2^2)$,

therefore $\qquad 3FK_4 = 6a_1 K_4 + 6K_2 K_3$,

therefore $\qquad FK_4 = 2a_1 K_4 + 2K_2 K_3$.

Similarly, we shall get

$$FK_5 = 2a_1 K_5 + 2K_2 K_4 + K_3^2.$$

Now, suppose $\quad FK_n = 2a_1 K_n + 2K_2 K_{n-1} + 2K_3 K_{n-2} + \ldots$,

where the terms after $2a_1 K_n$ have the same form as A_{n-3} in the operation O_3, viz.,

$$2a_0 a_{n-3} + 2a_1 a_{n-4} + \ldots .$$

Then $\qquad F\Gamma K_n = \Gamma FK_n$,

or $\qquad nFK_{n+1} = \Gamma(2a_1 K_n + 2K_2 K_{n-1} + \ldots)$.

Now Γ acting on a K-function is equivalent to

$$2K_3 \frac{d}{dK_2} + 3K_4 \frac{d}{dK_3} + \ldots,$$

by § 2, (2), so that its effect on

$$2K_2 K_{n-1} + \ldots$$

is the same as that of

$$2a_1 \frac{d}{da_0} + 3a_2 \frac{d}{da_1} + \ldots \text{ on } A_{n-3},$$

which, by § 1, (1) and (2), is

$$(n-2)\, A_{n-2} + 2A_{n-2} - 2a_0 a_{n-2} = nA_{n-2} - 2a_0 a_{n-2},$$

while the operation of Γ on $2a_1 K_n$ is

$$2na_1 K_{n+1} + 2K_2 K_n .$$

Hence $nFK_{n+1} = 2na_1 K_{n+1} + n$ (expression corresponding to A_{n-2}),

therefore $\qquad FK_{n+1} = 2a_1 K_{n-1} + 2K_2 K_n + 2K_3 K_{n-1} + \ldots .$

Thus we have proved inductively the general expression for the effect of F on K_n.

The result of the above investigation is, of course, that F will leave

unaltered any K-function (but for a power of $2a_1$) which has the form of a secondary invariant in K_2, K_3, K_4 .... Thus

$$F(K_2K_4-K_3^2) = 4a_1(K_2K_4-K_3^2),$$

$$F(K_2^2K_5-2K_2K_3K_4+K_3^3) = 6a_1(K_2^2K_5-2K_2K_3K_4+K_3^3).$$

Such invariants we may call ***ultra-secondary.***

5. Let us now consider the conditions for a K-function's being annihilable by O_1.

Now $$O_1\Gamma-\Gamma O_1$$

$$= \Sigma\left\{(n+1)(n+2)\,a_0a_n-(n+2)\,A_n-n\,(n+1)\,a_0a_n+nA_n\right\}\delta_n \text{ by § 1}$$

$$= 2\Sigma\left\{(n+1)\,a_0a_n-A_n\right\}\delta_n$$

$$= 2a_0(i+w)-2E = 4a_0i-2E \quad\text{.......................................(1).}$$

Now $O_1K_2 = 0$, by trial, and by (1),

$$O_1\Gamma K_2 = 8a_0K_2-2EK_2 = 2a_0K_2 \text{ by § 3,}$$

i.e., $$O_1K_3 = a_0K_2.$$

Again, $$O_1\Gamma K_3-\Gamma O_1K_3 = 12a_0K_3-2EK_3 = 4a_0K_3,$$

therefore $$3O_1K_4 = a_0\Gamma K_2+4a_0K_3 = 6a_0K_3,$$

therefore $$O_1K_4 = 2a_0K_3.$$

And, inductively, we get

$$O_1K_n = (n-2)\,a_0K_{n-1} \quad\text{.......................(2),}$$

so that the effect of O_1 on a K-function is the same as

$$a_0\left(K_2\frac{d}{dK_3}+2K_3\frac{d}{dK_4}+\dots\right).$$

Thus the condition required is that the K-function should have the form of a primary invariant. Such we may call ***ultra-primary.***

Collecting the results relating to K-functions, we see

(1) A general isobaric K-function is a secondary invariant, *i.e.*, is annihilated by O_2, and has two grade-equations, giving its degree and weight. See § 2.

(2) If the K-function is also homogeneous, the corresponding a-function has a third grade-equation giving the degree + weight of the K-function. See § 3.

(3) If the K-function is a primary invariant, so also is the a-function. See § 5.

(4) If the K-function is a secondary invariant, the a-function has a fourth grade-equation. See § 4.

The last three kinds we call *ultra-homogeneous*, *ultra-primary*, and *ultra-secondary*, respectively. We now come to the class of satisfied invariants for any order.

6. *Satisfied Secondary Invariants.*

Such invariants are to be annihilated by Ω_2 and by Ω_n which stands for

$$na_1\delta_0+(n-1)\,a_2\delta_1+(n-2)\,a_3\delta_2+\dots.$$

Hence

$$O_2\Omega_n-\Omega_n O_2=0$$

$$=(A_0\delta_1+A_1\delta_2+\dots)\left\{na_1\delta_0+(n-1)\,a_2\delta_1+\dots\right\}$$
$$-(na_1\delta_0+\dots)(A_0\delta_1+\dots)$$
$$=nA_0\delta_0+\left\{(n-1)\,A_1-\Omega_n A_0\right\}\delta_1+\left\{(n-2)\,A_2-\Omega_n A_1\right\}\delta_2+\dots$$

Now $\quad \Omega_n=(n+1)(a_1\delta_0+a_2\delta_1+\dots)-(a_1\delta_0+2a_2\delta_1+\dots),$

so that its operation on A_m gives, by § 1,

$$2\,(n+1)\,A_{m+1}-2\,(a_0n+1)\,a_0a_{m+1}-(m+1)\,A_{m+1}.$$

This makes the above annihilator reduce to

$$nA_0\delta_0+\left\{2\,(n+1)\,a_0a_1-(n+2)\,A_1\right\}\delta_1$$
$$+\left\{2\,(n+1)\,a_0a_2-(n+2)\,A_2\right\}\delta_2+\dots,$$

i.e., making the first term similar to the following,

$$\left\{2\,(n+1)\,a_0^2-(n+2)\,A_0\right\}\delta_0+\left\{2\,(n+1)\,a_0a_1-(n+2)\,A_1\right\}\delta_1+\dots=0,$$

i.e., $(n+2)\left\{A_0\delta_0+A_1\delta_1+A_2\delta_2+\dots\right\}=2a_0\,(n+1)(a_0\delta_0+a_1\delta_1+\dots),$

i.e., by § 3, $\quad E\equiv a_0^2\delta_0+2a_0a_1\delta_1+\dots=\dfrac{2a_0\,(n+1)\,i}{n+2}\quad\dots\dots\dots\dots\dots(1):$

a very remarkable equation, since it shows that, in a satisfied secondary invariant, $2i$ must be divisible by $(n+2)$, since the whole operation E is integral.

Such is found to be the case in all the catalecticants.

For instance, the effect of E on the third-degree invariant of a quartic is equivalent to multiplying it by

$$2a_0\times\frac{5\times3}{6}=5a_0,$$

as can easily be verified.

We see moreover, by § 3, that all satisfied secondaries are ultra-homogeneous.

7. We can obtain a series of other grade-equations in the following manner.

The alternant operation $E\Omega_n - \Omega_n E$

$$= (nA_1 - \Omega_n A_0)\,\delta_0 + \{(n-1)\,A_2 - \Omega_n A_1\}\,\delta_1 + \ldots$$

$$= (n+1)\,\{(2a_0a_1 - A_1)\,\delta_0 + (2a_0a_2 - A_2)\,\delta_1 + \ldots\} \text{ as in § 2}$$

$$= -(n+1)\,\{a_1^2\delta_1 + 2a_1a_2\delta_2 + (2a_1a_3 + a_2^2)\,\delta_3 + \ldots\}$$

$$= -(n+1)\,F, \text{ see § 4.}$$

Now, $E\Omega_n = 0$, obviously, while

$$\Omega_n E = (na_1\delta_0 + \ldots)\,\frac{2a_0\,(n+1)\,i}{n+2} \text{ by § 6}$$

$$= \frac{2a_1n\,(n+1)\,i}{n+2}.$$

Hence $$a_1^2\delta_1 + 2a_1a_2\delta_2 + \ldots = \frac{2a_1\,ni}{n+2} \quad\ldots\ldots\ldots\ldots(1).$$

From this we see that satisfied secondaries are also secondary invariants in the K's, *i.e.*, ultra-secondary.

Alternating this last operation with Ω_n, we get

$$a_2^2\delta_3 + 2a_2a_3\delta_3 + (2a_2a_4 + a_3^2)\,\delta_4 + \ldots = a_1^2\delta_0 + a_2\,\frac{2i\,(n-1)}{n+2},$$

and so on, but these grade-equations become too complicated to be of much interest.

In § 6, (1), let $a_0 = 0$. Then we see that the part of a secondary invariant annihilated by Ω_n independent of a_0 is annihilated by

$$a_1^2\delta_2 + 2a_1a_2\delta_3 + \ldots,$$

i.e., has itself the form of a secondary.

Thus the quartic catalecticant becomes

$$-(a_1^2a_4 - 2a_1a_2a_3 + a_2^3),$$

which answers to the secondary

$$a_0^2a_3 - 2a_0a_1a_2 + a_1^3,$$

Again, in (1), in this section, if $a_1 = 0$, we have

$$a_2^2\delta_3 + 2a_2a_3\delta_4 + \ldots = 0,$$

which shows that in a satisfied secondary, if $a_1 = 0$, all the coefficients of the powers of a_0 have the form of secondaries if the suffixes be reduced by 2. Thus the quartic catalecticant becomes

$$a_0(a_2a_4 - a_3^2) - a_2^3.$$

8. *Satisfied Double Invariants.*

From the relation § 6, (1), we shall be able to deduce the fact that the catalecticants are the only possible satisfied double invariants.

Let us consider what the highest power of a_0 will be in a secondary invariant J, where

$$(n+2)\,EJ = 2a_0\,(n+1)\,iJ.$$

Let $J = C_0a_0^h + \ldots$ terms containing lower powers of a_0, where C_0 and the other coefficients are independent of a_0. Now

$$E \equiv -a_0^2\delta_0 + 2a_0\,(a_0\delta_0 + a_1\delta_1 + a_2\delta_2 + \ldots)$$
$$+ a_1^2\delta_2 + \ldots \text{ terms not containing } a_0.$$

Operating on J, we get

$$-ha_0^{h+1}C_0 + 2ia_0^{h+1}C_0 + \ldots = \frac{2a_0\,(n+1)\,i}{n+2}\,a_0^{h+1}\,C_0 + \ldots$$

Equating coefficients of a_0^{h+1}, we see that

$$h = 2i - \frac{2\,(n+1)\,i}{n+2} = \frac{2i}{n+2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

By the symmetry of a satisfied invariant, we know that the highest power of a_n in a satisfied double invariant is also $\dfrac{2i}{n+2}$.

Now, let us consider what is the smallest power a_n can have in a satisfied invariant of order n and degree i. ($w = \tfrac{1}{2}ni$.)

Let J be the invariant, and written in full let it be $K_0a_n^h + \ldots$, where K_0 does not contain a_n.

Since $\quad (a_0\delta_1 + 2a_1\delta_2 + \ldots)(K_0a_n^h + \ldots) = 0$ identically,

we easily get that $\quad (a_0\delta_1 + 2a_1\delta_2 + \ldots)\,K_0 = 0,$

i.e., K_0 is a semi-invariant.

Moreover, since

$$(a_n\delta_{n-1}+2a_{n-1}\delta_{n-2}+\dots)(K_0a_n^h+\dots)=0,$$

we easily get $$\delta_{n-1}K_0=0,$$

i.e., K_0 is of order $n-2$. Its weight is, of course,

$$w-nh \text{ or } n(\tfrac{1}{2}i-h),$$

while its degree is $$i-h.$$

Now, in an unsatisfied invariant,

$$ni > 2w,$$

so that, in K_0, $$(n-2)(i-h)>n(i-2h),$$

an inequality which reduces to

$$h>\frac{2i}{n+2}\dots\dots\dots\dots(2),$$

except when $ni=2w$, when it becomes an equality.

Conversely, if $h=\dfrac{2i}{n+2}$, then $(n-2)(i-h)=n(i-2h)$, so that K_0 is a satisfied invariant.

We see, then, that a satisfied double invariant of order n and degree i has the minimum leading power of a_n and a_0.

Now, since $$(a_0^2\delta_1+2a_0a_1\delta_2+\dots)(K_0a_n^h+\dots)=0,$$

we easily get $$(a_0^2\delta_1+2a_0a_1\delta_2+\dots)K_0=0,$$

so that K_0 is a satisfied double invariant of order $n-2$ and degree $i-h=\dfrac{ni}{n+2}$.

Hence the highest power of a_{n-2} in K_0 must also be the minimum,

i.e., $$\frac{2(i-h)}{n}=\frac{2i}{n+2}, \text{ i.e., } h.$$

Moreover, there is no a_{n-3} in the leading term, and similarly the power of a_{n-4} occurring in it must also be h.

By this inductive method we see eventually that the leading term is $(a_na_{n-2}\dots a_0)^h$, which necessitates that n be even.

Now by symmetry the minimum power of a_0 in a satisfied invariant must also be the $\left(\dfrac{2i}{n+2}\right)^{\text{th}}$, and such can only occur when n is even.

Now we know of the existence of one such invariant for order n, namely the catalecticant, in which a_0 occurs in the first power. Let us write this Ka_0+L, where K and L are independent of a_0, and suppose that J is another invariant with minimum leading power in a_0, which expanded in powers of a_0 is

$$K^\lambda a_0^\lambda + K_1 a_0^{\lambda-1} + \dots.$$

Then $\qquad J-(Ka_0+L)^\lambda$, *i.e.*, $(K_1-\lambda K^{\lambda-1}L)\,a_0^{\lambda-1}+\dots$,

is also an invariant of same degree as J, but with leading power of a_0 equal to $(\lambda-1)$, *i.e.*, less than the minimum, which is impossible. It is, moreover, impossible to reduce $K_1-\lambda K^{\lambda-1}L$ to the necessary form $kK^{\lambda-1}$, for k would have to be a factor running through the whole derived invariant, and hence be itself an invariant. But this is impossible, since it is independent of a_0.

From this we see that, excepting the catalecticants, there are no satisfied invariants of any given order which have the leading power of a_0 a minimum. Hence there are no satisfied double invariants except the catalecticants.

We can now express the catalecticants as K-functions. We easily get, by direct calculation, that

$$a_0^3 \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = (K_2K_4-K_3^2),$$

which verifies the results of the previous sections. For, since the catalecticant is a double invariant (see Mr. Forsyth's paper, *Messenger*, Feb. 1888), and is known to be satisfied for order 4, we know that E and F are grade-operations, so that the right side would be homogeneous (§ 3), and a secondary invariant (§ 4), and a primary (§ 5).

Similarly the catalecticant,
$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{vmatrix},$$

multiplied by some power of a_0, must = some primary (§ 5), homogeneous (§ 3), secondary (§ 4), function of the K's, of which the leading term must be $K_2K_4K_6$, § 2, (3). This function must be that similar to the quartic catalecticant after reducing the suffixes by two, and we easily see (since $w=\frac{1}{2}ni$) that it is satisfied for degree 4.

Hence

$$a_0^8 \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{vmatrix} = \begin{vmatrix} K_2 & K_3 & K_4 \\ K_3 & K_4 & K_5 \\ K_4 & K_5 & K_6 \end{vmatrix}.$$

Similarly the catalecticant

$$\begin{vmatrix} K_2 & & & \\ & K_4 & & \\ & & \ddots & \\ & & & K_{2n} \end{vmatrix} = a_0^{n^2-1} \begin{vmatrix} a_0 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{2n+2} \end{vmatrix}.$$

9. *Other Generators.*

There are other generators of secondary invariants, but perhaps the most remarkable is

$$a_0(a_1\delta_0 + 2a_2\delta_1 + 3a_3\delta_2 + \dots) - \frac{a_1}{a_0}E,$$

which, it may be observed, is integral, since the result of operating with E always brings in a factor a_0 (see §3). Let us call this generator L.

To prove its generative properties, we have, as in §2, to show that

$$(LO_2 - O_2L)J = 0 \quad \text{if} \quad O_2J = 0.$$

Now

$$LO_2 - O_2L = \Sigma\left\{a_0(a_1\delta_0 + \dots)A_{n-1} - \frac{a_1}{a_0}EA_{n-1}\right.$$

$$\left. - a_0O_2(n+1)a_{n+1} + \frac{a_1}{a_0}OA_n\right\}\delta_n + a_0E$$

$$= \Sigma\{a_0 . nA_n - a_0(n+1)A_n + a_0A_n\}\delta_n - \frac{a_1}{a_0}\Sigma\{EA_{n-1} - OA_n\}\delta_n$$

$$= -\frac{a_1}{a_0}\Sigma\begin{Bmatrix} 2A_0a_{n-1} + 2A_1a_{n-2} + \dots & 2A_{n-1}a_0 \\ -2A_0a_{n-1} - \dots & 2A_{n-1}a_0 \end{Bmatrix}\delta_n$$

$$= 0,$$

therefore L is a generator.

The remarkable property of L is that it will produce the same set of protomorphs as Γ although algebraically independent of Γ. This will depend on the fact that the alternation of L and E reproduces L,

i.e.,

$$LE-EL=\Sigma\left\{a_0(a_1\delta_0+\ldots)A_n-\frac{a_1}{a_0}EA_n-E(n+1)a_0a_{n+1}+E\frac{a_1}{a_0}A_n\right\}\delta_n$$

$$=\Sigma\left\{(n+1)a_0A_{n+1}-\frac{a_1}{a_0}EA_n-(n+1)a_0^2a_{n+1}-(n+1)a_0A_{n+1}\right.$$

$$\left.+2a_1A_n-a_1A_n+\frac{a_1}{a_0}EA_n\right\}\delta_n$$

$$=\Sigma\left\{a_1A_n-(n+1)a_0^2A_{n+1}\right\}\delta_n=-a_0L.$$

Hence $$EL-LE=a_0L.$$

For instance, $ELK_2=LEK_2+a_0LK_2=4a_0LK_2$, by § 3.

Now we know LK_2 is a secondary invariant, and therefore can be expressed as a K-function. Moreover, since $ELK_2=4a_0LK_2$, we know it must be homogeneous in the K's. But, by inspecting the actual effect of L on K_2, we see that LK_2 must be of degree and weight 3, and its leading term is $3a_0^2a_3$.

Hence $LK_2=3K_3+$ other homogeneous terms. But it is obviously impossible to have other terms. Therefore

$$LK_2=3K_3.$$

Similarly $LK_3=4K_4+$ homogeneous and isobaric terms,

therefore $$LK_3=4K_4,$$

and generally $$LK_n=(n+1)K_{n+1}\quad\ldots\ldots\ldots\ldots(1).$$

From this we deduce the equation

$$(L-\Gamma)K_n=K_{n+1},$$

which leads us to a proof of the interesting fact that all the terms in K_{n+1} are integral. For $L-\Gamma$ is a numerically integral operator, and consequently K_3, K_4 ... are all integral, *i.e.*, every term in ΓK_n is divisible by n, and every term in LK_n by $(n+1)$, see § 2 (2), and § 9 (1).

The actual calculation of the protomorphs or of any homogeneous function of the K's can be easily calculated by means of the formula $E=a_0(W+I)$, where $W=$ weight and $I=$ degree of the K-function,

so that $$W=w=i.$$

For $$E=-a_0^2\delta_0+2a_0i+a_1^2\delta_2+2a_1a_2\delta_3+(2a_1a_3+a_2^2)\delta_4+\ldots$$

$$=-a_0^2\delta_0+2a_0i+M\text{ say.}$$

Let the K function $= C_0 a_0^m + C_1 a_0^{m-1} + C_2 a_0^{m-2} + \dots$.

Then $\quad (-a_0^2\delta_0 + 2a_0 i + M)(C_0 a_0^m + \dots) = a_0 (W+I)(C_0 a_0^m + \dots)$

therefore
$$MC_0 a_0^m + MC_1 a_0^{m-1} + \dots$$
$$-ma_0^{m+1} C_0 - (m-1)\, a_0^m C_1 - \dots$$
$$+(2i - W - I)(a_0^{m+1} C_0 + \dots) = 0,$$

$$\therefore \quad m = 2i - W - I = i - I, \qquad \therefore \text{ degree of } C_0 \text{ is } I.$$

$$MC_0 = \{(m-1)-(2i-W-I)\}\, C_1 = -C_1,$$
$$MC_1 = -2C_2,$$
$$MC_2 = -3C_3,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

whence the K function

$$= (\epsilon^{-M/a_0})\, C_0 a_0^m \text{ where } C_0 \text{ is the leading coefficient.}$$

Thus
$$K_2 = \epsilon^{-M/a_0} a_0 a_2 = a_0 a_2 - a_1^2,$$
$$K_3 = \epsilon^{-M/a_0} a_0^2 a_3 = a_0^2 a_3 - a_0 M a_3 + \tfrac{1}{2} M^2 a_3$$
$$= a_0^2 a_3 - 2a_0 a_1 a_2 + a_1^3,$$
$$\&c., \qquad \&c.,$$

while
$$K_2 K_4 - K_3^2 = \epsilon^{-M/a_0} a_0^4 (a_2 a_4 - a_3^2)$$
$$= a_0^4 (a_2 a_4 - a_3^2) - a_0^3 (a_1^2 a_4 + 2a_1 a_2 a_3 + a_2^3 - 4a_1 a_2 a_3)$$
$$= a_0^3 (a_0 a_2 a_4 - a_0 a_3^2 - a_1^2 a_4 + 2a_1 a_2 a_3 - a_2^3),$$

as we have already seen.

The most interesting examples of ultra-homogeneous secondaries are the catalecticants, which, by applying the above formula, give the relation

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3, & . & . \\ a_1, & a_2, & a_3, & . & . & . \\ a_2, & a_3, & . & . & & \\ a_3, & . & . & & & \end{vmatrix} = \epsilon^{-M/a_0} a_0 \begin{vmatrix} a_2, & a_3, & . \\ a_3, & . & \\ . & & \end{vmatrix}$$

whence it follows that

$$(a_1^2\delta_2+2a_1a_2\delta_3+\ldots)\begin{vmatrix} a_2, & a_3, & a_4 \\ a_3, & a_4, & \cdot \\ a_4, & \cdot & \end{vmatrix}$$

$$=-\begin{vmatrix} 0, & a_1, & a_2, & a_3 \\ a_1, & a_2, & a_3, & \\ a_2, & a_3, & & \\ a_3, & & & \end{vmatrix}$$

for any degree whatever; an identity which is interesting in its generality.

It is to be noticed that all ultra-homogeneous secondaries have a covariantive form. In fact, if we depress every suffix by unity, and put x for a_0, we get a quantic in x,

$$C_0x^m-C_1x^{m-1}+C_2x^{m-2}-\ldots,$$

such that
$$O_2C_0=C_1,$$
$$O_2C_1=2C_2, \text{ \&c.},$$

i.e., we get a secondary covariant.

It may be useful to notice the similarities and dissimilarities between secondary invariants and their cognate functions, primary invariants and pure reciprocants.

In all three classes of functions we get a system of eductive protomorphs, but only in secondary invariants do we get two generators yielding the same set of protomorphs.

In all three classes we have two grade-equations, one for the degree and the other for the weight, but only for certain secondaries is there a third grade-equation E. Of course there are primaries with this property, viz., the double invariants, but they are only a subdivision of secondaries, and I have not been able to discover any pure reciprocants with an analogous property.

The existence of these ultra-homogeneous forms seems to me to be the most interesting fact in the theory of secondaries, but I have not yet been able to discover any internal property of such invariants, *i.e.*, any property belonging to their complete primitives. The complete primitives of the protomorphs are not easy to obtain.

$$K_2=0 \text{ gives } y(ax+b)=cx+d,$$

$$K_3=0 \text{ gives } Ay+B=\tan(Cx+D),$$

$$K_2K_4-K_3^2=0 \text{ gives } y(ax^2+bx+c)=dx^2+ex+f,$$

and K_4 depends on elliptic functions; but, beyond these and the other catalecticants, I have not been able to get any complete primitives of ultra-homogeneous invariants.

10. Double invariants may be looked upon as a class of algebraic invariants, annihilated by a second operation O_2.

Suppose, for instance, the sextic

$$a_0x^6+6a_1x^5y+15a_2x^4y^2+30a_3x^3y^3+15a_4x^2y^4+6a_5xy^5+a_6y^6$$
$$=\beta_1(x-\alpha_1y)^6+\beta_2(x-\alpha_2y)^6+\beta_3(x-\alpha_3y)^6+\beta_4(x-\alpha_4y)^6.$$

Then, by evaluating a_0, a_1, ... in terms of β's and α's, we easily get the identical relation

$$\frac{a_0}{z}+\frac{a_1}{z^2}+\frac{a_2}{z^3}+\ldots=\frac{\beta_1}{z-\alpha_1}+\frac{\beta_2}{z-\alpha_2}+\frac{\beta_3}{z-\alpha_3}+\frac{\beta_4}{z-\alpha_4}=u \text{ say.}$$

Now a change of a_1 into $a_1+\lambda A_0$, a_2 into $a_2+\lambda A_1$, &c., denotes a change of u into $u+\lambda u^2$, or $\dfrac{u}{1-\lambda u}$, where u is very small, *i.e.*,

$$\frac{\beta_1(z-\alpha_2)(z-\alpha_3)\ldots+\ldots}{(z-\alpha_1)(z-\alpha_2)\ldots} \text{ into } \frac{\beta_1(z-\alpha_2)(z-\alpha_3)\ldots+\ldots}{(z-\alpha_1)(z-\alpha_2)\ldots-\lambda\beta_1(z-\alpha_2)\ldots}$$
$$=\frac{\text{same numerator}}{(z-\alpha_1-\lambda\beta_1)(z-\alpha_2-\lambda\beta_2)\ldots},$$

which may be shown to be equivalent to changing

$$\alpha_r \text{ into } \alpha_r+\lambda\beta_r,$$

and $$\beta_r \text{ into } \beta_r+2\lambda\beta_r\left\{\frac{\beta_1}{\alpha_1-\alpha_r}+\frac{\beta_2}{\alpha_2-\alpha_r}+\ldots\right\},$$

i.e., the change in the a's can be expressed as a definite change in the α's and β's, and is independent of the number of α's and β's.

For instance, the condition for reducing a sextic to the sum of three sixth powers will be a secondary invariant, for then

$$\frac{a_0}{z}+\frac{a_1}{z^2}+\ldots=\frac{\beta_1}{z-\alpha_1}+\frac{\beta_2}{z-\alpha_2}+\frac{\beta_3}{z-\alpha_3}.$$

Changing a_1 in $a_1+\lambda A_0$, &c., we still get on the left side three fractions similar to those we had before, so that we can still reduce the sextic to three sixth powers. Hence the condition is not altered by changing a_1 into $a_1+\lambda A_0$, &c., *i.e.*, it is annihilable by O_2. It is, in fact, the equating to zero of the catalecticant, which we have already seen is a secondary invariant.

Thursday, March 14th, 1889.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Mr. C. E. Haselfoot, B.A., Fellow of Hertford College, Oxford, was elected a member, and Messrs. Roseveare and W. W. Taylor were admitted into the Society.

The following papers were read:—

Notes on Plane Curves, (iv.) Involution-condition of a Cubic and its Hessian; (v.) Figure of a certain Cubic and its Hessian: the President (Mr. Elliott in the chair).

On Play "à outrance": Major P. A. MacMahon, R.A.

Some Results in the Elementary Theory of Numbers: C. Leudesdorf, M.A.

The Characteristics of an Asymmetric Optical Combination: Dr. J. Larmor, M.A.

A new Angular and Trigonometrical Notation, with applications H. MacColl, M.A.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLV., Nos. 275 and 276.

"Educational Times," for March.

"Transactions of the Cambridge Philosophical Society," Vol. XIV., Part III.

"Transactions of the Royal Irish Academy," Vol. XXIX., Part V.

"The Collected Mathematical Papers of Arthur Cayley, Sc.D., F.R.S.," Vol. I.,* 4to; Cambridge, 1889.

"Annals of Mathematics," Vol. IV., No. 4.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," Tome III., 20 Cahier, 1887.

"Bulletin de la Société Mathématique de France," Tome XVI., No. 6.

"Observations Pluviométriques et Thermométriques faites dans le Département de la Gironde de Juin, 1886, à Mai, 1887, Note de M. Rayet;" 8vo pamphlet; Bordeaux, 1887.

"Rendiconti del Circolo Matematico di Palermo," Tomo III., Fasc. 1, Jan., Feb., 1889.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIII., Stück 2.

"Nieuw Archief voor Wiskunde," Deel XV., Stuk 2.

"Acta Mathematica," XII., 2.

"Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2°, Vol. II., Fasc. 1—12.

"Atti della Reale Accademia dei Lincei — Rendiconti," Vol. IV., Fasc. 10; Roma, 1888.

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* The Society also subscribe for a copy of these papers.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich," Vol. xxxiii., No. 2.

"Bollettino delle Pubblicazioni Italiane, ricevute par Diritto di Stampa," Nos. 75 and 76; Index to Ditto for 1888 (3 parts).

"The Planets upon Cardioides," by the Rev. G. T. Carruthers, M.A., Chaplain of Subatha, India.

"The Cause of Light," by the same.

"Supposed First Ascertainment of the Area of a Spherical Triangle by the Infinitesimal Analysis," by John Spare, M.A., M.D., New Bedford, Mass. (2 pp.).

The Characteristics of an Asymmetric Optical Combination.

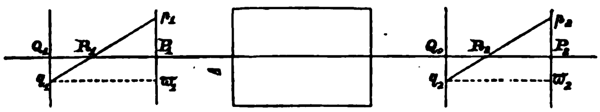
By J. Larmor.

[*Read March 14th*, 1889.]

1. The general properties of optical combinations which are symmetrical round an axis, such as ordinary telescopes and microscopes, have, as is well known, been analysed by Gauss; and the performance of these instruments, when aberrations are left out of account, has been shown to depend simply on three constants—which may be taken geometrically as the coordinates of the two principal points and the two principal foci, between whose mutual distances one linear relation exists.

It is now customary, and it conduces to clearness of view, to throw the theory into a geometrical form in the manner first completely set forth by Maxwell (*Quarterly Journal of Mathematics*, II, 1858). To effect this, without entering into details of the construction of the special instrument, we are confined to the use of only those properties that are *characteristic* of rays of light in general. For the purposes of the problem, when restricted to symmetry round the axis, these properties may be stated in the simple approximate form that (i) all rays proceeding from a point go to form the image of that point, (ii) for all such rays the time of passage from point to image ($\Sigma\mu\delta s$) is the same, because they belong to the same wave spreading out from the object point, and finally, after passing through the instrument, converging to the image. The application of the former of these principles requires that the rays are everywhere inclined at a small angle to the axis, as is usually the case in practice.

2. It will be convenient to begin by briefly analysing the parts played by these two fundamental principles in the theory; the following mode of procedure is simple and comprehensive.



Let P_1, P_2 and Q_1, Q_2 be two pairs of conjugate foci on the axis, and let the linear magnification transverse to the axis (appropriately called simply the *magnification*) of small objects at P_1, Q_1 be m_p, m_q, respectively; these are clearly the same in all azimuths round the axis. Draw any ray $q_1 R_1 p_1$ meeting the planes through P_1, Q_1, transverse to the axis in p_1, q_1, and meeting the axis in R_1; after passing through the instrument let its path be $q_2 R_2 p_2$. Then, by the principle of images, $Q_2 q_2 = m_q \,.\, Q_1 q_1$, $P_2 p_2 = m_p \,.\, P_1 p_1$ (if the image were inverted, the sign of m would be negative); so that

$$\frac{Q_2 R_2}{R_2 P_2} = \frac{m_q}{m_p} \cdot \frac{Q_1 R_1}{R_1 P_1} \quad \text{......................... (1).}$$

This law of simple proportions determines absolutely the relative positions of conjugate foci R_1, R_2. To find the corresponding magnification m_r, it is only necessary to draw a ray through Q_1, meeting the transverse planes through R_1, P_1 in ρ_1, ϖ_1, and passing out at the other side through Q_2, ρ_2, ϖ_2. Then

$$m_r = \frac{R_2 \rho_2}{R_1 \rho_1} = m_p \, \frac{Q_2 R_2}{Q_2 P_2} \Big/ \frac{Q_1 R_1}{Q_1 P_1} \quad \text{.................... (2).}$$

The principle of images is thus sufficient to determine completely the performance of the instrument.

3. But hitherto no notice has been taken of the nature of the optical media in which the object and image lie. We may introduce this consideration by aid of the second general principle. It can be applied by comparing two similarly situated rays, which have therefore the same value of $\Sigma \mu \delta s$ in passage through the instrument. Two such are the rays through q_1, one, $q_1 p_1$, through R_1, the middle point of $Q_1 P_1$, and the other $q_1 \varpi_1$, parallel to the axis. The expression $\Sigma \mu \delta s$ has the same value S from p_1 to p_2 as from ϖ_1 to ϖ_2, because these points are symmetrically situated above and below the axis. Hence, considering the two rays from q_1 to q_2, we have

$$\mu_1 \cdot q_1 p_1 + S - \mu_2 \cdot q_2 p_2 = \mu_1 \cdot q_1 \varpi_1 + S - \mu_2 \cdot q_2 \varpi_2,$$

therefore $\qquad \mu_1(q_1p_1-q_1\varpi_1) = \mu_2(q_2p_2-q_2\varpi_2).$

Therefore, approximately, since the transverse distances are supposed small compared with the longitudinal, and $P_1p_1 = P_1\varpi_1$, we have

$$\mu_1\frac{4Q_1q_1^2}{2Q_1P_1} = \mu_2\frac{(Q_2q_2+P_2p_2)^2-(Q_2q_2-P_2\varpi_2)^2}{2Q_2P_2},$$

$$\mu_1\frac{4}{Q_1P_1} = \mu_2\frac{(m_q+m_p)^2-(m_q-m_p)^2}{Q_2P_2},$$

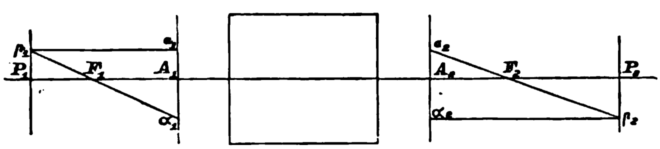
$$\frac{Q_2P_2}{Q_1P_1} = \frac{\mu_2}{\mu_1}.m_pm_q \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The results (1) and (2) give the position and transverse magnification of the image of a small object anywhere situated. The result (3) forms a very elegant expression (given by Maxwell) for the longitudinal magnification (termed by Maxwell the *elongation*) of any object not confined to be small, viz., it is equal to the ratio of the indices multiplied by the product of the transverse magnifications of its extremities.

It follows from (3) that $m_p = \left(\frac{\mu_2\alpha_2}{\mu_1\alpha_1}\right)^{-1}$, where α_1, α_2 are the inclinations of the ray to the axis.

It is also clear that (3) includes (2), so that (1) and (3) form a complete system of fundamental formulæ. The points P_1, Q_1 may be taken as the origins of measurement; and then three observations suffice to determine all the constants of the instrument.

4. The scheme introduced by Gauss makes use of the *principal points* A_1, A_2, for which $m = +1$, and of the principal foci F_1, F_2.



The image of a point p_1 may now be constructed by drawing rays p_1a_1 parallel to the axis, and $p_1F_1\alpha_1$ through F_1, and tracing them on from their points of emergence on the other principal plane to their point of intersection at p_2. Since

$$A_1a_1 = A_2a_2, \quad A_1\alpha_1 = A_2\alpha_2,$$

(because $m = +1$) it follows that the instrument behaves as if the principal planes through A_1, A_2 were coincident, if we leave out of

account the shifting of the image system along the axis consequent on moving A_2 into coincidence with A_1. With the exception of this shifting, the instrument therefore behaves as a single thin lens whose principal foci are at distances from it equal to A_1F_1, A_2F_2. In fact it follows in a well-known manner from the diagram that, if we consider lines measured from a principal point or focus as positive when away from the instrument,

$$F_1P_1 \,.\, F_2P_2 = F_1A_1 \,.\, F_2A_2 = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots (4),$$

the generalisation of Newton's formula for a lens; which also leads to

$$\frac{A_1P_1}{A_1F_1} + \frac{A_2P_2}{A_2F_2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The principle $\Sigma\mu\delta s =$ constant for all rays from p_1 to p_2 requires in the same way as above that

$$\frac{A_1F_1}{\mu_1} = \frac{A_2F_2}{\mu_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Thus when $\mu_1 = \mu_2$, $\quad A_1F_1 = A_2F_2 = f$,

where f may be called the focal length of the instrument; for it is the focal length of the *simple equivalent lens*, convex when f is positive, which, as we have seen, is equivalent to the instrument in all respects except as regards the situation of the image system along the axis. The instrument has, in this case, an *optical centre*, which is the middle point of A_1A_2.

5. To determine experimentally the constants of the instrument, we may proceed by any of the known methods that apply to lenses. (i) The positions of the principal foci F_1, F_2 may be marked by the aid of a parallel beam; then the positions of any pair of conjugate foci yield the value of f^2 by formula (4). To determine whether the positive or negative value of f is to be taken, we must observe whether the image is inverted or erect. (ii) The bright point may be moved along a graduated scale till its conjugate focus approaches it most nearly; the distance between them is then A_1A_2+4f, and the centre of the instrument is equidistant from them. Another observation will complete the determination. This method requires modification if A_1A_2+4f is small or negative. (iii) Two conjugate foci at a distance c apart being selected, the instrument is shifted along its axis through a distance a till they again become conjugate; then, if $A_1A_2 = b$,

$$a^2 = (c-b)(c-b-4f).$$

Another observation will suffice.

6. With an instrument (telescope) so focussed that a system of rays parallel to the axis emerges parallel to the axis, the magnification is obviously constant at all distances. Therefore so also is the elongation, by (3). The value of the magnification is now the essential constant of the instrument; the exact position of the image system along the axis being, as before, a thing inessential.

7. This brief sketch has shown that a symmetrical optical combination may usually be replaced by its simple equivalent lens, so far as regards the relative position and magnitudes of the images formed. It is easy to see that it may be replaced by two lenses so as to give their *exact* positions on the axis.

The principal motive of this discussion is to inquire how far similar specifications hold for instruments not symmetrical.

The properties *characteristic* of a ray system now take a more complicated form, viz., that elaborated by Sir W. Rowan Hamilton, from the fundamental property that $\Sigma\mu\delta s$ is stationary along a ray, between any two points, whether in the first medium, or in the final medium, or situated in any other way, and that it has the same value, proportional to the time of propagation, for all the rays from one wave front to another.

The general problem before us is, given the path of the central ray of a pencil of light which traverses any system of media which may be heterogeneous and may be doubly-refracting, but passes from an initial homogeneous isotropic medium to a final one of the same character, to determine how many and what kind of observations in the initial and final media would be necessary in order to obtain a complete account of the nature of the change produced by transmission through the system in any pencil proceeding along this path.

The method of discussing the propagation of a narrow pencil on Hamilton's principles has been set set forth afresh by Maxwell (*Proc. Lond. Math. Soc.*, VI, 1874 and 1875); in what follows we shall use a similar analysis.

8. Take origins O_1, O_2 on the central ray in the initial and final media, with axes Z_1, Z_2 tangential to the ray and directed *both away from* the refracting system. Let the value of the reduced path $(\Sigma\mu\delta s)$ for a ray between points $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ in the transverse planes through these origins be

$$U \equiv \text{const.} + \tfrac{1}{2}a_1x_1^2 + c_1x_1y_1 + \tfrac{1}{2}b_1y_1^2 + px_1x_2 + qx_1y_2 + rx_2y_1 + sy_1y_2$$
$$+ \tfrac{1}{2}a_2x_2^2 + c_2x_2y_2 + \tfrac{1}{2}b_2y_2^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

As U is stationary near the axis, because the transverse planes are tangential to the wave fronts, there can be no terms of the first degree in it.

The ten constants in its expression may be called the constants of the optical combination, when referred to these origins.

9. The first aim must therefore be to choose new origins which will reduce this expression to its simplest form. Taking then the planes $Z_1 = \gamma_1^{-1}$, $Z_2 = \gamma_2^{-1}$ as new transverse planes, we have for the reduced distance between the points $(\xi_1, \eta_1, \gamma_1^{-1})$, $(\xi_2, \eta_2, \gamma_2^{-1})$ in these planes the expression

$$V = \mu_1 s_1 + U + \mu_2 s_2 \quad\text{..........................(8)},$$

where
$$s_1 = \{\gamma_1^{-2} + (x_1 - \xi_1)^2 + (y_1 - \eta_1)^2\}^{\frac{1}{2}}$$

$$= \gamma_1^{-1} + \tfrac{1}{2}\gamma_1 (x_1^2 + y_1^2) - \gamma_1 (x_1 \xi_1 + y_1 \eta_1) + \tfrac{1}{2}\gamma_1 (\xi_1^2 + \eta_1^2) \quad\text{......}(9_1),$$

$$s_2 = \gamma_2^{-1} + \tfrac{1}{2}\gamma_2 (x_2^2 + y_2^2) - \gamma_2 (x_2 \xi_2 + y_2 \eta_2) + \tfrac{1}{2}\gamma_2 (\xi_2^2 + \eta_2^2) \quad\text{......}(9_2).$$

We have to eliminate x_1, y_1 and x_2, y_2 from V. Now, since $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ are in the free path of this ray, we must have

$$\frac{\partial V}{\partial x_1} = 0, \quad \frac{\partial V}{\partial y_1} = 0, \quad \frac{\partial V}{\partial x_2} = 0, \quad \frac{\partial V}{\partial y_2} = 0;$$

therefore
$$\left.\begin{aligned}
(\mu_1\gamma_1 + a_1)\, x_1 + c_1 y_1 + p x_2 + q y_2 &= \mu_1 \gamma_1 \xi_1 \\
c_1 x_1 + (\mu_1 \gamma_1 + b_1)\, y_1 + r x_2 + s y_2 &= \mu_1 \gamma_1 \eta_1 \\
p x_1 + r y_1 + (\mu_2 \gamma_2 + a_2)\, x_2 + c_2 y_2 &= \mu_2 \gamma_2 \xi_2 \\
q x_1 + s y_1 + c_2 x_2 + (\mu_2 \gamma_2 + b_2)\, y_2 &= \mu_2 \gamma_2 \eta_2
\end{aligned}\right\} \quad\text{...........(10)}.$$

These equations determine the paths in the initial and final media of the ray which goes from $(\xi_1, \eta_1, \gamma_1^{-1})$ to $(\xi_2, \eta_2, \gamma_2^{-1})$.

10. By solving these equations (10) we obtain the coordinates of the points in which the ray from $(\xi_1, \eta_1, \gamma_1^{-1})$ to $(\xi_2, \eta_2, \gamma_2^{-1})$ meets the transverse planes through the origins. If, however, the determinant Δ of their left-hand sides vanishes, the system is equivalent to only three independent equations, together with the condition

$$A_1 \mu_1 \gamma_1 \xi_1 + C_1 \mu_1 \gamma_1 \eta_1 + P \mu_2 \gamma_2 \xi_2 + Q \mu_2 \gamma_2 \eta_2 = 0 \quad\text{...........(11)},$$

in which A_1, C_1, P, Q are the minors of the first column (or any other column) of the determinant. The three independent equations do not now determine the coordinates x_1, y_1 in terms of $(\xi_1, \eta_1, \gamma_1^{-1})$ and $(\xi_2, \eta_2, \gamma_2^{-1})$, but merely lead to a linear relation between these coor-

dinates, of the form

$$Sx_1 + Qy_1 = \text{constant} \quad\quad\quad (12),$$

where Q, S are the minors of the terms q, s in the determinant Δ.

Also $$Qx_2 + Py_2 = \text{constant}.$$

The interpretation is clearly as follows. By (12) the rays from $(\xi_1, \eta_1, \gamma_1^{-1})$ to $(\xi_2, \eta_2, \gamma_2^{-1})$ now form a singly infinite system, which cut the transverse plane through O_1 along a line, and similarly cut the transverse through O_2 along a line. For different pairs of points subject to the relation (11), on the same transverse planes $Z_1 = \gamma_1^{-1}$, $Z_2 = \gamma_2^{-1}$, these lines form parallel systems. The meaning of this relation (11) itself is, that to the point $(\xi_1, \eta_1, \gamma_1^{-1})$ there correspond a system of points which form a line in the plane $Z_2 = \gamma_2^{-1}$; and that, corresponding to different points in the plane $Z_1 = \gamma_1^{-1}$, these lines are all parallel.

The positions of the two transverse planes $Z_1 = \gamma_1^{-1}$, $Z_2 = \gamma_2^{-1}$ are connected by the relation $\Delta = 0$, which is quadratic in both γ_1 and γ_2. Thus for a beam proceeding from any point in one of these planes, the *focal lines* lie in the two conjugate planes determined by the equation $\Delta = 0$. For different points in the same transverse plane γ_1 the focal lines lie in the same pair of conjugate planes γ_2, and form two parallel systems, each inclined to the corresponding axis x_2 at an angle whose tangent is $(-Q/P)$. For all points in either of the planes γ_2, one of the focal lines lies in the plane γ_1, and is inclined to the axis x_1 at an angle whose tangent is $(-S/Q)$; and this line is one of a parallel system in that plane, such that all points on one of them have the same focal line in the plane γ_2.

11. The equation (11) now shows that all the rays emanating from a luminous line or slit whose equation is

$$\mu_1 \gamma_1 (A_1 \xi_1 + C_1 \eta_1) = H \quad\quad\quad (13),$$

pass through the line whose equation is

$$\mu_2 \gamma_2 (P\xi_2 + Q\eta_2) = -H \quad\quad\quad (14).$$

This remark has an interesting bearing on the working of refracting spectroscopes. It shows, in fact, that without any special adjustment of the refracting prismatic surfaces the slit may always be rotated into such a position that its image for any monochromatic light will be a sharp line. But it will not be an image in the ordinary sense of corresponding point for point with the slit. This position of the slit is the one inclined to the axis of x, at an angle

whose tangent is $(-A_1/C_1)$, in which A_1, C_1 are expressed as above in terms of the position of the image, and the constants of the combination.

12. To return to § 9, by solving (10) and substituting in the expression for V, we obtain the function characteristic of the combination when referred to the new transverse planes. The work is simplified by making use of the fact that V is a minimum as above, so that

$$\frac{\partial V}{\partial x_1}=0,\ \ \frac{\partial V}{\partial x_2}=0,\ \ldots$$

By Euler's theorem of homogeneous functions

$$V=\text{const.}+\mu_1 s_1+\mu_2 s_2+2\left(\frac{\partial U}{\partial x_1}x_1+\frac{\partial U}{\partial x_2}x_2+\frac{\partial U}{\partial y_1}y_1+\frac{\partial U}{\partial y_2}y_2\right),$$

where

$$\frac{\partial U}{\partial x_1}=\frac{\partial V}{\partial x_1}-\mu_1\gamma_1(x_1-\xi_1),$$

$$\frac{\partial U}{\partial x_2}=\frac{\partial V}{\partial x_2}-\mu_2\gamma_2(x_2-\xi_2),$$

$$\frac{\partial U}{\partial y_1}=\frac{\partial V}{\partial y_1}-\mu_1\gamma_1(y_1-\eta_1),$$

$$\frac{\partial U}{\partial y_2}=\frac{\partial V}{\partial y_2}-\mu_2\gamma_2(y_2-\eta_2).$$

Therefore

$$\begin{aligned}V&=\text{const.}+\mu_1 s_1+\mu_2 s_2-\tfrac{1}{2}\mu_1\gamma_1x_1(x_1-\xi_1)-\tfrac{1}{2}\mu_1\gamma_1y_1(y_1-\eta_1)\\&\qquad-\tfrac{1}{2}\mu_2\gamma_2x_2(x_2-\xi_2)-\tfrac{1}{2}\mu_2\gamma_2y_2(y_2-\eta_2)\\&=\text{const.}+\mu_1\gamma_1^{-1}-\tfrac{1}{2}\mu_1\gamma_1(\xi_1x_1+\eta_1y_1)+\tfrac{1}{2}\mu_1\gamma_1(\xi_1^2+\eta_1^2)\\&\qquad+\mu_2\gamma_2^{-1}-\tfrac{1}{2}\mu_2\gamma_2(\xi_2x_2+\eta_2y_2)+\tfrac{1}{2}\mu_2\gamma_2(\xi_2^2+\eta_2^2)\ \ldots\ldots\ldots(15).\end{aligned}$$

In this expression, which is remarkable as not involving explicitly any of the constants, the values of x_1, y_1, ... in terms of ξ_1, η_1, ... are to be substituted.

13. Before proceeding with this substitution it will be convenient to examine how the coefficients p, q, r, s may be simplified by rotation of the axes. Changing to polar coordinates by writing

$$\left.\begin{aligned}x_1&=\rho_1\cos\theta_1\\y_1&=\rho_1\sin\theta_1\end{aligned}\right\},\qquad\left.\begin{aligned}x_2&=\rho_2\cos\theta_2\\y_2&=\rho_2\sin\theta_2\end{aligned}\right\},$$

we have $\qquad px_1x_2+qx_1y_2+rx_2y_1+sy_1y_2$

$$= \tfrac{1}{2}\rho_1\rho_2 \left\{(p+s)\cos\chi-(q-r)\sin\chi+(p-s)\cos\psi+(q+r)\sin\psi\right\},$$

where $\qquad \psi = \theta_1+\theta_2, \quad \chi = \theta_1-\theta_2,$

$$= \tfrac{1}{2}\rho_1\rho_2 \left\{P\cos(\chi+\epsilon)+P'\cos(\psi+\epsilon')\right\} \quad \text{............(16).}$$

The expressions $\qquad P^2 = (p+s)^2+(q-r)^2,$

$$P'^2 = (p-s)^2+(q+r)^2,$$

are therefore invariant for the transformation; that is,

$$\text{(i)} \quad p^2+s^2+q^2+r^2 \quad \text{..........................(17),}$$

$$\text{(ii)} \quad ps-qr \quad \text{..................................(18),}$$

are invariant.

We cannot therefore make p, q, r, s all vanish. It is possible to have $q = 0$, $r = 0$; and then $p = s$ will involve a transformation of origins such that the first invariant (i) is twice the second (ii).

Conversely this latter condition by itself gives

$$(p-s)^2+(q+r)^2 = 0,$$

so that, as the quantities are to remain real, it amounts to two conditions $p = s$, $q = -r$, whatever be the directions of the axes.

14. We proceed to work out the transformation to new origins. It is easy to see that

$$V = \text{const.}+\tfrac{1}{2}\mu_1\gamma_1(\xi_1^2+\eta_1^2)+\tfrac{1}{2}\mu_2\gamma_2(\xi_2^2+\eta_2^2)+\tfrac{1}{2}W \quad \text{......(19),}$$

where

$$\begin{vmatrix} \mu_1\gamma_1+a_1 & c_1 & p & q \\ c_1 & \mu_1\gamma_1+b_1 & r & s \\ p & r & \mu_2\gamma_2+a_2 & c_2 \\ q & s & c_2 & \mu_2\gamma_2+b_2 \end{vmatrix} W$$

$$= \begin{vmatrix} \mu_1\gamma_1+a_1 & c_1 & p & q & \mu_1\gamma_1\xi_1 \\ c_1 & \mu_1\gamma_1+b_1 & r & s & \mu_1\gamma_1\eta_1 \\ p & r & \mu_2\gamma_2+a_2 & c_2 & \mu_2\gamma_2\xi_2 \\ q & s & c_2 & \mu_2\gamma_2+b_2 & \mu_2\gamma_2\eta_2 \\ \mu_1\gamma_1\xi_1 & \mu_1\gamma_1\eta_1 & \mu_2\gamma_2\xi_2 & \mu_2\gamma_2\eta_2 & \cdot \end{vmatrix} \quad \text{......(20).}$$

The coefficients of the terms

$$p'\xi_1\xi_2+q'\xi_1\eta_2+r'\xi_2\eta_1+s'\eta_1\eta_2$$

on the right-hand side are twice the corresponding minors in Δ, the coefficient of W, multiplied by $\mu_1\gamma_1\mu_2\gamma_2$; they involve, if common multipliers are laid aside, only γ_1, γ_2, and $\gamma_1\gamma_2$, and these linearly.

We notice that

$$(p's'-q'r')\,\Delta = (ps-qr)(\mu_1\gamma_2\mu_2\gamma_2)^2.$$

The conditions $p'=s'$, $q'=-r'$, lead to a quadratic equation to determine γ_1, and then a linear equation for γ_2. If the roots of the quadratic are real, there are determined in this way two pairs of principal points on the central ray. We cannot then, by turning both sets of axes through the same angle round that ray, make $q'=0$, so that also $r'=0$ by the relation of invariance in § 13; but this can be done by rotation through different angles.

The condition for the reality of these principal points is that a certain function of the coefficients, which it is unnecessary to write out, should be positive.

When this condition is not satisfied, we might attempt to make the reduction in another way. By turning the axes through different angles and moving the origins, we can make $p'=s'$, $q'=0$, $r'=0$ in an infinite number of ways. The considerations now to be given will show that this process leads to the same result as the above.

The case of imaginary roots may, however, be reduced to the real one by the addition of a known astigmatic lens or obliquely-placed simple lens, orientated to the proper azimuth, whose presence may be afterwards allowed for.

15. Resuming the original notation, there are therefore, with the above restriction, two pairs of origins and axes for either of which the function characteristic of the combination assumes the form

$$\begin{aligned}U \equiv \text{const.} &+ \tfrac{1}{2}a_1x_1^2+c_1x_1y_1+\tfrac{1}{2}b_1y_1^2\\ &+p\,(x_1x_2+y_1y_2)\\ &+\tfrac{1}{2}a_2x_2^2+c_2x_2y_2+\tfrac{1}{2}b_2y_2^2+\dots \qquad \dots(21).\end{aligned}$$

This may in a manner be taken as the canonical form of the characteristic.

We can now express very simply in Maxwell's manner the equations which determine the form of the emergent beam. Let the characteristic functions of the same beam in the neighbourhoods of

the two origins O_1, O_2 be

$$V_1 = \mu_1 z_1 + \tfrac{1}{2}A_1 x_1^2 + C_1 x_1 y_1 + \tfrac{1}{2}B_1 y_1^2 + \dots \quad \dots\dots\dots(22_1),$$

$$V_2 = \mu_2 z_2 + \tfrac{1}{2}A_2 x_2^2 + C_2 x_2 y_2 + \tfrac{1}{2}B_2 y_2^2 + \dots \quad \dots\dots\dots(22_2).$$

The other possible quadratic terms are absent because we must have identically satisfied the characteristic equation,

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = \mu^2 \quad \dots\dots\dots\dots\dots\dots(23)$$

The value of the reduced path from ξ_1, η_1, ζ_1 to ξ_2, η_2, ζ_2 is

$$\begin{aligned} V = &- (\mu_1 \zeta_1 + \tfrac{1}{2}A_1 \xi_1^2 + \dots) + \mu_1 z_1 + \tfrac{1}{2}A_1 x_1^2 + \dots \\ &+ U + \mu_2 z_2 + \tfrac{1}{2}A_2 x_2^2 + \dots - (\mu_2 \zeta_2 + \tfrac{1}{2}A_2 \xi_2^2 + \dots) \quad \dots\dots\dots(24), \end{aligned}$$

in which, as V is stationary, we must have

$$\frac{\partial V}{\partial x_1} = 0, \quad \frac{\partial V}{\partial x_2} = 0, \quad \frac{\partial V}{\partial y_1} = 0, \quad \frac{\partial V}{\partial y_1} = 0,$$

so that

$$\left.\begin{aligned} (a_1 + A_1)\, x_1 + (c_1 + C_1)\, y_1 + p x_2 &= 0 \\ (c_1 + C_1)\, x_1 + (b_1 + B_1)\, y_1 + p y_2 &= 0 \\ p x_1 + (a_2 + A_2)\, x_2 + (c_2 + C_2)\, y_2 &= 0 \\ p y_2 + (c_2 + C_2)\, x_2 + (b_2 + B_2)\, y_2 &= 0 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots(25).$$

These equations determine the point x_2, y_2 which corresponds to any point x_1, y_1, in two ways. On solving the first pair for x_1, y_1, and comparing with the second pair, there result the relations,

$$\frac{a_2 + A_2}{b_1 + B_1} = \frac{b_2 + B_2}{a_1 + A_1} = -\frac{c_2 + C_2}{c_1 + C_1} = \frac{p^2}{\Delta'} \quad \dots\dots\dots\dots\dots\dots(26),$$

where $\quad \Delta' = (a_1 + A_1)(b_1 + B_1) - (c_1 + C_1)^2.$

These equations determine the constants of the emergent beam.

When the incident pencil comes from a focus, $A_1 = B_1$, $C_1 = 0$. It is easy to verify that $(A_2 - B_2)/C_2$ is constant for all positions of that focus, only provided

$$(a_2 - b_2)/c_2 = (a_1 - b_1)/c_1;$$

it is only under this condition that the focal lines corresponding to all foci are parallel.

16. By changing U to polar coordinates, we see that the conditions that the combination is symmetrical round the axis, or acts as such a one, are $a_1 = b_1$, $c_1 = 0$, $a_2 = b_2$, $c_2 = 0$.

If we add to the instrument a thin astigmatic lens at O_1, of index μ and thickness t_1, whose effective thickness $(\mu-\mu_1)\,t_1$ is equal to

$$\tfrac{1}{2}\,(a_1+\lambda_1)\,x_1^2+c_1x_1y_1+\tfrac{1}{2}\,(b_1+\lambda_1)\,y_1^2,$$

and a thin astigmatic lens at O_2 whose effective thickness $(\mu-\mu_2)\,t_2$ is equal to

$$\tfrac{1}{2}\,(a_2+\lambda_2)\,x_2^2+c_2x_2y_2+\tfrac{1}{2}\,(b_2+\lambda_2)\,y_2^2,$$

the total resulting combination will be given by

$$U = \text{const.}-\lambda_1\,(x_1^2+y_1^2)+p\,(x_1x_2+y_1y_2)-\lambda_2\,(x_2^2+y_2^2)\ldots\ldots(27),$$

and therefore it will act as if it were symmetrical round the axis.

A pencil from O_1 as focus will not be affected by the first lens. It will therefore reach the second lens with a circular cross section if it has started with a circular section in the first medium, and has not been so wide as to be partially stopped out somewhere in the instrument. The (so-called) circle of least confusion of the emergent pencil will therefore be at the second lens.

Now it is to be remarked that a pencil of light rays, though it always passes through two focal lines, does not in general possess a circular cross-section at any point on its course. It is evident in fact that the condition for its having a circular section is that it should diverge symmetrically from each focal line.

Hence to find the position of O_1, place a stop in the path of the incident beam so as to make it circular; and move along the axis the luminous point which emits it until the emergent beam is symmetrical with respect to either of its focal lines; as may be tested by focussing a telescope on the line, covering it with a cross-wire, and then putting the telescope out of focus without rotating the wire. The position of O_2 is then at the corresponding circular cross-section of the emergent beam; or it may be similarly determined by placing the luminous point on the other side of the instrument. The result incidentally appears that if a conical pencil from O_2 has a circular cross-section at O_1,* then a conical pencil from O_1 has a circular section at O_2.

Find now by Prof. Stokes'† or any other method, the astigmatic lens

* This theorem of reciprocity may be proved directly from consideration of the general function of $x_1y_1z_1$ and $x_2y_2z_2$ characteristic of the combination. It is a case of the proposition formulated and discussed in Thomson and Tait's *Natural Philosophy*, §§ 334, 335.

† *Brit. Assoc. Report*, 1849, p. 10; *Collected Papers*, Vol. II., p. 172.

which placed in the final medium at O_2, would convert a parallel beam into the actual emergent beam ; this will be a lens whose principal focal lengths are equal and of opposite signs to the distances of the focal lines from the circular cross-section of the pencil, measured in the direction the light is travelling.

Proceed similarly with light emitted from the point O_2, and determine the corresponding lens in the medium at O_1.

The combination is equivalent to these two lenses, together with an instrument symmetrical with respect to the axis, whose principal foci are at O_1, O_2. The principal point P_1 of the latter on the side O_1 is at once determined as the point at which a transverse plane must be placed that the beam from O_1 may mark out on it a circle equal to its circular cross-section at O_2 on emergence; and in the same way the other principal point P_2 may be determined,—or else by the relation $O_1P_1/\mu_1 = O_2P_2/\mu_2$.

17. The experimental determinations here sketched amount to the specification of an instrument which is equivalent to the combination; that is, they give the optical constants of the combination.

The special case in which O_1 or O_2 is at an infinite distance may be noticed. The effect of the lens at O_1 is then to alter a pencil coming from a focus to one coming from two focal lines in fixed directions at constant distances from that focus.

A more special case still would be that of a *quasi*-telescope for which all parallel incident pencils emerge parallel.

18. The positions of these principal points O_1, O_2 become indeterminate, in the case of an instrument with a straight axis, when the planes of the principal curvatures of the pair of astigmatic lenses are parallel; and in the more general case, when c_1 and c_2 vanish or can be made to vanish by rotating the axes on both sides through the same angle. This leads to the condition

$$(a_1-b_1)/c_1 = (a_2-b_2)/c_2,$$

as at the end of § 15.

This is the single condition necessary that the combination should behave as one having a straight axis and symmetrical with respect to two perpendicular planes through the axis. In the general notation of § 9, it is therefore the condition that $P/Q =$ constant, when $\Delta = 0$.

When the instrument is of this simple character the course of any ray may be constructed by finding those of its traces on the two pairs of corresponding principal planes in the initial and final media, passing

through the axis of the combination. For it is clear, as Maxwell remarks, from the form

$$U = \text{const.} + \tfrac{1}{2}a_1x_1^2 + px_1x_2 + \tfrac{1}{2}a_2x_2^2$$
$$+ \tfrac{1}{2}b_1y_1^2 + sy_1y_2 + \tfrac{1}{2}b_2y_2^2 \quad \text{..............(27)},$$

that in the determination of the path of a ray by the method of § 15, the terms containing x_1, x_2 are now separated from the terms containing y_1, y_2.

Therefore the projections of any ray on the planes of x_1z_1 and x_2z_2 are now related by a construction similar to that which applies to instruments symmetrical round an axis, *i.e.*, there are two pairs of *quasi*-principal foci in these planes, and two pairs of *quasi*-principal points, by means of which these projections may be constructed. And having thus obtained the traces of a given emergent ray on the planes x_2z_2, y_2z_2, in terms of those of the same ray when incident on the planes x_1z_1, y_1z_1, the problem is solved.

This also follows more directly from the remark that the instrument is essentially the same as a straight one with the principal planes of its two terminal aplanatic lenses parallel, and therefore planes of symmetry.

It remains to show how these planes, and their cardinal points, may be identified experimentally. All rays incident in the plane x_1z_1 emerge in the plane x_2z_2. Therefore the plane x_2z_2 must be that containing the axis z_2, and a focal line corresponding to any focus on the axis z_1; there are therefore two such planes, which are the same whatever point on the axis z_1 is taken for focus, as has been already seen. In the same way the two planes x_1z_1 may be found by taking an incident focus on the axis z_2. These planes may be grouped into the two corresponding pairs in obvious ways. The positions of the cardinal points in a corresponding pair of planes may be determined by methods exactly analogous to those briefly sketched in § 5. The optical combination in question will then have been completely explored.

It might at first sight be imagined that the four possible adjustments of the axes would be sufficient to make c_1, c_2, q, r vanish always, and so reduce every combination to this type. But from the above, these conditions must be inconsistent with each other, unless a special relation between the constants is satisfied.

On Play "à outrance." *By* Major P. A. MacMahon, R.A.

[*Read March 14th*, 1889.]

Suppose two players A and B to have a and b counters respectively and to play a number of games, the winner receiving a counter from the loser until one of the players is deprived of all of his counters. We may consider the probability of all of the counters remaining finally with A, being given the probability of his winning any one game. When the probability of A winning each game is equal to a constant p, and no game can be drawn, the problem has been solved. (See Todhunter's *History of the Theory of Probability.*) I introduce here the consideration of the effect of supposing that the probability of a player winning a certain game depends in some manner upon the number of counters which he possesses at the commencement of that game.

In regard to the game which is commenced when the players A and B have respectively x and $n-x$ counters (where $a+b=n$), let the chances of winning be as $\phi(x)$ to $\phi(n-x)$, where $\phi(x)$ is a known function of x, which is finite for all integer values of the argument from 1 to n. Further, let u_x denote the chance of A winning the "partie" when he is in possession of x counters at the commencement of a game.

The chance of A winning the next game is then

$$\frac{\phi(x)}{\phi(x)+\phi(n-x)},$$

and the chance of his losing it is

$$\frac{\phi(n-x)}{\phi(x)+\phi(n-x)};$$

hence the difference equation

$$u_x = \frac{\phi(x)}{\phi(x)+\phi(n-x)}\,u_{x+1} + \frac{\phi(n-x)}{\phi(x)+\phi(n-x)}\,u_{x-1},$$

which may be written $\quad \dfrac{\phi(x)}{\phi(n-x)} = \dfrac{u_x-u_{x-1}}{u_{x+1}-u_x},$

and thence we obtain

$$\frac{\phi(n-1)}{\phi(1)}\cdot\frac{\phi(n-2)}{\phi(2)}\cdots\cdots\frac{\phi(n-s)}{\phi(s)} = \frac{u_{n-s}-u_{n-s-1}}{u_n-u_{n-1}}.$$

Put
$$1+\frac{\phi(n-1)}{\phi(1)}+\frac{\phi(n-1)\,\phi(n-2)}{\phi(1)\,\phi(2)}+\dots$$
$$+\frac{\phi(n-1)\,\phi(n-2)\dots\phi(n-s)}{\phi(1)\,\phi(2)\dots\phi(s)}=\Phi(s+1),$$

so that
$$\frac{u_n-u_{n-s-1}}{u_n-u_{n-1}}=\Phi(s+1),$$

and thence the two results,
$$\frac{u_n-u_x}{u_n-u_{n-1}}=\Phi(n-x),$$
$$\frac{u_n-u_0}{u_n-u_{n-1}}=\Phi(n);$$

from these, by division, $\dfrac{u_n-u_x}{u_n-u_0}=\dfrac{\Phi(n-x)}{\Phi(n)}$.

The conditions of the problem are
$$u_n=1,\quad u_0=0;$$

hence
$$1-u_x=\frac{\Phi(n-x)}{\Phi(n)},$$

and since, by definition,
$$\Phi(n-x)+\Phi(x)=\Phi(n),$$

we obtain
$$u_x=\frac{\Phi(x)}{\Phi(n)};$$

this is the solution of the difference equation subject to the given conditions; that is to say, the probability u_x is the ratio of the sum of the first x terms to the sum of all of the terms of the series—
$$1+\frac{\phi(n-1)}{\phi(1)}+\frac{\phi(n-1)\,\phi(n-2)}{\phi(1)\,\phi(2)}+\dots+\frac{\phi(n-1)}{\phi(1)}+1.$$

This reminds one of the binomial series,
$$(1+1)^{n-1},$$

with which it becomes identical when
$$\phi(x)=x.$$

When, in general,
$$\phi(x)=x^m,$$

where m may be any number, positive, zero, or negative, commensurable or not, we are concerned with powers of binomial coefficients.

If we write
$$\frac{\phi(x)}{\phi(x)+\phi(n-x)}=f(x),$$
so that
$$f(x)+f(n-x)=1,$$
the difference equation becomes
$$u_x-u_{x-1}=f(x)(u_{x+1}-u_{x-1}),$$
and the solution may be written,
$$u_x=\frac{F(x)}{F(n)},$$
where
$$F(x)=1+\frac{f(n-1)}{f(1)}+\frac{f(n-1)f(n-2)}{f(1)f(2)}+\dots$$
$$\dots+\frac{f(n-1)f(n-2)\dots f(n-x+1)}{f(1)f(2)\dots f(x-1)}.$$

I now suppose that when A and B have respectively x and $n-x$ counters the chances of winning the next game are as $\phi(x)$ to $f(n-x)$, the most general case.

The difference equation becomes
$$u_x=\frac{\phi(x)}{\phi(x)+f(n-x)}u_{x+1}+\frac{f(n-x)}{\phi(x)+f(n-x)}u_{x-1},$$
or
$$\frac{\phi(x)}{f(n-x)}=\frac{u_x-u_{x-1}}{u_{x+1}-u_x},$$
putting
$$x=n-s,$$
$$\frac{\phi(n-s)}{f(s)}=\frac{u_{n-s}-u_{n-s-1}}{u_{n-s+1}-u_{n-s}},$$
thence
$$\frac{\phi(n-1)\phi(n-2)\dots\phi(n-s)}{f(1)f(2)\dots f(s)}=\frac{u_{n-s}-u_{n-s-1}}{u_n-u_{n-1}},$$
put
$$\left(\frac{\phi}{f}\right)_{s+1}=1+\frac{\phi(n-1)}{f(1)}+\frac{\phi(n-1)\phi(n-2)}{f(1)f(2)}+\dots$$
$$\dots+\frac{\phi(n-1)\dots\phi(n-s)}{f(1)\dots f(s)},$$
and then
$$\frac{u_n-u_{n-s-1}}{u_n-u_{n-1}}=\left(\frac{\phi}{f}\right)_{s+1},$$
which gives
$$\frac{u_n-u_x}{u_n-u_{n-1}}=\left(\frac{\phi}{f}\right)_{n-x},$$
and
$$\frac{u_n-u_0}{u_n-u_{n-1}}=\left(\frac{\phi}{f}\right)_n,$$

and remembering that $u_n = 1$, $u_0 = 0$, we obtain, by division,

$$1 - u_x = \frac{\left(\frac{\phi}{f}\right)_{n-x}}{\left(\frac{\phi}{f}\right)_n},$$

or

$$u_x = \frac{\left(\frac{\phi}{f}\right)_x}{\left(\frac{\phi}{f}\right)_n},$$

showing that to obtain u_x we have to divide the sum of the first x terms of the series,

$$\left(\frac{\phi}{f}\right)_n,$$

by the series itself.

Similarly, if v_x be the chance that B wins the partie when he has x counters,

$$v_x = \frac{\left(\frac{f}{\phi}\right)_x}{\left(\frac{f}{\phi}\right)_n},$$

and since

$$u_x + v_{n-x} = 1,$$

we obtain the interesting identity,

$$\frac{\left(\frac{\phi}{f}\right)_x}{\left(\frac{\phi}{f}\right)_n} + \frac{\left(\frac{f}{\phi}\right)_{n-x}}{\left(\frac{f}{\phi}\right)_n} = 1,$$

and the value of u_x may be presented in either of the forms shown by

$$u_x = \frac{\left(\frac{\phi}{f}\right)_x}{\left(\frac{\phi}{f}\right)_n} = \frac{\left(\frac{f}{\phi}\right)_n - \left(\frac{f}{\phi}\right)_{n-x}}{\left(\frac{f}{\phi}\right)_n}.$$

This result may be of service when it is necessary to take account of the courage of the two players. Thus putting

$$f(x) = px^{\alpha},$$

$$\phi(x) = qx^{-\beta},$$

where α and β are small quantities determined from experience, we would have the circumstance of A playing better when he is winning and of B playing better when he is losing, and doubtless a more correct estimate of the probability of either player winning the partie *would thus be* formed.

Some Results in the Elementary Theory of Numbers.

By C. LEUDESDORF, M.A.

[*Read March* 14*th*, 1889.]

It is well known that if p is a prime number, all the coefficients of the equation

$$(x-1)(x-2)(x-3) \ldots (x-p+1)+1 = 0$$

are divisible by p, and that consequently the sum S_μ of the μ^{th} powers of the numbers 1, 2, 3, ... $p-1$ is divisible by p unless μ is a multiple of $p-1$ (see, *e.g.*, Serret, *Algèbre Supérieure*, Vol. II., p. 46). And it is clear in the same way that if $S_{-\mu}$, the sum of the inverse μ^{th} powers of the same numbers, be formed, the result of the addition will be a fraction whose numerator is divisible by p unless μ is a multiple of $p-1$. Since all the factors of the denominator of such fraction are prime to p, we may, for shortness, say that $S_{-\mu}$ is divisible by p unless μ is a multiple of $p-1$.

But this does not exhaust the whole truth. In fact, if μ is odd, S_μ and $S_{-\mu}$ are both divisible by p^2 with certain exceptions. In particular the sum of the reciprocals of 1, 2, 3, ... $p-1$ is a fraction whose numerator is always divisible by p^2 except when $p=2$ or $p=3$.* Results similar in character hold when p is replaced by any composite number N, and the numbers 1 to $p-1$ by the numbers less than N and prime to it. The object of the present paper is to prove the foregoing statements, and to examine into the divisibility by the various prime factors of any number N of the sum of the μ^{th} powers of the numbers less than N and prime to it. I confine myself chiefly to the inverse powers, as the results when μ is positive are of less interest and present less difficulty, since the actual value of S_μ can be found by summing the series.

1. Here, and in all that follows, μ will be supposed to be an *odd number*.

Using $M(p)$ to denote a multiple of p, and $\psi_\mu(p)$ to represent

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* The remark that if the reciprocals of 1, 2, 3, ... $p-1$ are added together, the numerator of the resulting fraction is generally divisible not only by p, but by p^2, when p is a prime, was made to me some little time since by Mr. J. M. Dyer, M.A., of Eton College; and it is this remark which first led me to the subject of the present paper. I desire also to record my indebtedness to Mr. Morgan Jenkins, M.A., for some emendations and suggestions, especially in connection with § 2, which have proved very useful.

what has been hitherto called $S_{-\mu}$, we have

$$\psi_\mu(p) = \frac{1}{1^\mu} + \frac{1}{2^\mu} + \frac{1}{3^\mu} + \dots + \frac{1}{(p-1)^\mu}$$

$$= \tfrac{1}{2}\Sigma\left\{\frac{1}{x^\mu} + \frac{1}{(p-x)^\mu}\right\} \text{ from } x=1 \text{ to } x=p-1$$

$$= \frac{p}{2}\Sigma_1^{p-1}\frac{(p-x)^{\mu-1}-(p-x)^{\mu-2}x+\dots+x^{\mu-1}}{x^\mu(p-x)^\mu} \text{ since } \mu \text{ is odd}$$

$$= \frac{p}{2}\left\{M(p) - \frac{\mu}{2}\Sigma_1^{p-1}\frac{x^{\mu-1}}{x^{2\mu}}\right\}$$

$$= \tfrac{1}{2}M(p^2) - \frac{\mu p}{2}\psi_{\mu+1}(p).$$

Therefore (excluding only the case $p = 2$) it is seen that $\psi_\mu(p)$ is $M(p^2)$ if $\frac{\mu}{2}\psi_{\mu+1}(p)$ is $M(p)$, which is the case unless $\mu+1 = M(p-1)$ and at the same time μ is not $M(p)$. Now if $\mu+1 = M(p-1)$ and $\mu = M(p)$ at the same time, μ must be of the form $p\{(p-1)t-1\}$. The conclusion is, therefore, that if μ is odd, $\psi_\mu(p)$ is in general divisible by p^2; but if $\mu+1 = M(p-1)$, then $\psi_\mu(p)$ is only divisible by p, except in the cases where μ is of the form $p\{(p-1)t-1\}$, when $\psi_\mu(p)$ is divisible by p^2 as in general.

Exceptions.—If $p = 2$, $\psi_\mu(p)$ reduces to 1 and is not divisible by p at all, as is indeed shown by the formula at once.

And if $p = 3$, $\mu+1$, being even, must be $M(p-1)$; therefore $\psi_\mu(3)$ is only divisible by 3^2 when μ is an odd multiple of 3, and in all other cases $\psi_\mu(3)$ is divisible by 3, but not by 3^2.

2. Let now N be any number, and let $f_\mu(N, n)$ denote the sum of the inverse μ^{th} powers of the numbers, prime to N, which lie between nN and $(n+1)N$, where μ is odd as before. Then, since if $nN+x$ is such a number, $(n+1)N-x$ is another,

$$f_\mu(N, n) = \tfrac{1}{2}\Sigma\left\{\frac{1}{(nN+x)^\mu} + \frac{1}{(nN+N-x)^\mu}\right\},$$

the summation taking place for every positive integral value of x less than N and prime to N; and the coefficient $\frac{1}{2}$ being required because $nN+x$ and $nN+N-x$ are repeated and interchanged when x is changed to $N-x$. Since μ is odd, $f_\mu(N, n)$ is evidently divisible by

$nN+x+nN+N-x$, that is, by $(2n+1)N$. So

$$f_\mu(N,0) = \tfrac{1}{2}\Sigma\left\{\frac{1}{x^\mu}+\frac{1}{(N-x)^\mu}\right\},$$

therefore $$\frac{1}{2n+1}f_\mu(N,n)-f_\mu(N,0)$$

$$=\tfrac{1}{2}\Sigma\left\{\frac{(nN+x)^{-\mu}+(nN+N-x)^{-\mu}}{2n+1}-\frac{x^{-\mu}+(N-x)^{-\mu}}{1}\right\}$$

$$=\tfrac{1}{2}\Sigma x^{-\mu}\left\{\frac{\left(1+\frac{nN}{x}\right)^{-\mu}+(-1)^\mu\left(1-\overline{n+1}\frac{N}{x}\right)^{-\mu}}{2n+1}-\frac{1+(-1)^\mu\left(1-\frac{N}{x}\right)^{-\mu}}{1}\right\}$$

$$=\Sigma\tfrac{1}{2}x^{-\mu}\left\{\frac{\left[1-\mu n\frac{N}{x}+\frac{\mu(\mu+1)}{1.2}n^2\frac{N^2}{x^2}-\dots\right]-\left[1+\mu(n+1)\frac{N}{x}+\frac{\mu(\mu+1)}{1.2}(n+1)^2\frac{N^2}{x^2}+\dots\right]}{2n+1}\right.$$
$$\left.-1+\left[1+\mu\frac{N}{x}+\frac{\mu(\mu+1)}{1.2}\frac{N^2}{x^2}+\dots\right]\right\}.$$

The constant term inside the brackets vanishes, as also does the coefficient of $\frac{N}{x}$; the coefficient of the general term $\left(\frac{N}{x}\right)^{m+1}$ being

$$-\frac{\mu(\mu+1)\dots(\mu+m)}{1.2.3\dots(m+1)}\left\{\frac{(n+1)^{m+1}-(-n)^{m+1}}{2n+1}-1\right\}.$$

It is easily seen that this coefficient is always divisible by $n(n+1)$, and that $(n+1)^{m+1}-(-n)^{m+1}$ is always divisible by $2n+1$.

We have then

$$\frac{1}{2n+1}f_\mu(N,n)-f_\mu(N,0)=-\tfrac{1}{2}\Sigma x^{-\mu}\left\{\frac{\mu(\mu+1)(\mu+2)}{1.2.3}\frac{N^3}{x^3}(n^2+n)\right.$$
$$\left.+\frac{\mu(\mu+1)(\mu+2)(\mu+3)}{1.2.3.4}\frac{N^4}{x^4}2(n^2+n)+\dots\right\}$$

$$=-\frac{N^3}{2}(n^2+n)\frac{\mu(\mu+1)(\mu+2)}{1.2.3}\left\{\Sigma x^{-(\mu+3)}+\frac{\mu+3}{2}N\Sigma x^{-(\mu+4)}+\dots\right\}$$

$$=-\frac{N^3}{2}(n^2+n)\frac{\mu(\mu+1)(\mu+2)}{6}\left\{f_{\mu+3}(N,0)\right.$$
$$\left.+\frac{\mu+3}{2}Nf_{\mu+4}(N,0)+\dots\right\}\dots\dots\dots\dots(1).$$

3. Next, let $\psi_\mu(N)$ denote the sum of the inverse μ^{th} powers of all the numbers less than N, and prime to it [so that $\psi_\mu(N)$ is the same thing as $f_\mu(N, 0)$]. Further, let $a, b, c, \ldots$ be the prime factors of N, so that $N = a^l b^m c^n \ldots$ say, where $l, m, n, \ldots$ are supposed each greater than unity; and write $\dfrac{N}{a}$ for $a^{l-1} b^m c^n \ldots$. Then, since any number prime to N is prime to $\dfrac{N}{a}$,

$$\psi_\mu(N) = f_\mu\left(\frac{N}{a}, 0\right) + f_\mu\left(\frac{N}{a}, 1\right) + f_\mu\left(\frac{N}{a}, 2\right) + \ldots + f_\mu\left(\frac{N}{a}, a-1\right)$$

$$= f_\mu\left(\frac{N}{a}, 0\right)$$

$$+3\left[f_\mu\left(\frac{N}{a}, 0\right) - 1\,.\,2\,\frac{\mu(\mu+1)(\mu+2)}{12}\,\frac{N^3}{a^3}\left\{f_{\mu+3}\left(\frac{N}{a}, 0\right)\right.\right.$$
$$\left.\left. + \frac{\mu+3}{2}\,\frac{N}{a}\,f_{\mu+4}\left(\frac{N}{a}, 0\right) + M\left(\frac{N^2}{a^2}\right)\right\}\right]$$

$$+5\left[f_\mu\left(\frac{N}{a}, 0\right) - 2\,.\,3\,\frac{\mu(\mu+1)(\mu+2)}{12}\,\frac{N^3}{a^3}\left\{f_{\mu+3}\left(\frac{N}{a}, 0\right)\right.\right.$$
$$\left.\left. + \frac{\mu+3}{2}\,\frac{N}{a}\,f_{\mu+4}\left(\frac{N}{a}, 0\right) + M\left(\frac{N^2}{a^2}\right)\right\}\right]$$

$+$ &c.

$$+(2a-1)\left[f_\mu\left(\frac{N}{a}, 0\right) - (a-1)\,a\,\frac{\mu(\mu+1)(\mu+2)}{12}\,\frac{N^3}{a^3}\left\{f_{\mu+3}\left(\frac{N}{a}, 0\right)\right.\right.$$
$$\left.\left. + \frac{\mu+3}{2}\,\frac{N}{a}\,f_{\mu+4}\left(\frac{N}{a}, 0\right) + M\left(\frac{N^2}{a^2}\right)\right\}\right]$$

by equation (1) of § 2;

$$= (1+3+5+\ldots+\overline{2a-1})\,\psi_\mu\left(\frac{N}{a}\right)$$
$$-(1\,.\,2\,.\,3 + 2\,.\,3\,.\,5 + \ldots + \overline{2a-1}\,.\,\overline{a-1}\,.\,a)\,\frac{\mu(\mu+1)(\mu+2)}{12}$$
$$\left\{\frac{N^3}{a^3}\,\psi_{\mu+3}\left(\frac{N}{a}\right) + \frac{\mu+3}{2}\,M\left(\frac{N^4}{a^4}\right)\right\}$$

$$= a^2\psi_\mu\left(\frac{N}{a}\right) - \frac{\mu(\mu+1)(\mu+2)}{24}\,a^2(a^2-1)\left\{\frac{N^3}{a^3}\,\psi_{\mu+3}\left(\frac{N}{a}\right)\right.$$
$$\left. + \frac{\mu+3}{2}\,M\left(\frac{N^4}{a^4}\right)\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

$$\therefore\ \psi_\mu(N)-a^2\psi_\mu\left(\frac{N}{a}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}(a^2-1)$$

$$\times\left\{a^{3l-1}b^{3m}c^{2n}\dots\psi_{\mu+3}\left(\frac{N}{a}\right)+\frac{\mu+3}{2}M(a^{4l-2}b^{4m}c^{4n}\dots)\right\}.$$

Similarly,

$$\psi_\mu\left(\frac{N}{a}\right)-a^2\psi_\mu\left(\frac{N}{a^2}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}(a^2-1)$$

$$\times\left\{a^{3l-4}b^{3m}c^{3n}\dots\psi_{\mu+3}\left(\frac{N}{a^2}\right)+\frac{\mu+3}{2}M(a^{4l-6}b^{4m}c^{4n}\dots)\right\},$$

and so on, until finally

$$\psi_\mu\left(\frac{N}{a^{l-2}}\right)-a^2\psi_\mu\left(\frac{N}{a^{l-1}}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}(a^2-1)$$

$$\times\left\{a^5b^{3m}c^{3n}\dots\psi_{\mu+3}\left(\frac{N}{a^{l-1}}\right)+\frac{\mu+3}{2}M(a^6b^{4m}c^{4n}\dots)\right\},$$

from which, by multiplying by 1, a^2, a^4, ... a^{2l-4}, and adding

$$\psi_\mu(N)-a^{2l-2}\psi_\mu\left(\frac{N}{a^{l-1}}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}(a^2-1)$$

$$\times\left\{M(a^{2l+1}b^{3m}c^{3n}\dots)+\frac{\mu+3}{2}M(a^{2l+2}b^{4m}c^{4n}\dots)\right\}.$$

Similarly,

$$\psi_\mu\left(\frac{N}{a^{l-1}}\right)-b^{2m-2}\psi_\mu\left(\frac{N}{a^{l-1}b^{m-1}}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}(b^2-1)$$

$$\times\left\{M(a^3b^{2m+1}c^{3n}\dots)+\frac{\mu+3}{2}M(a^4b^{2m+2}c^{4n}\dots)\right\};$$

$$\therefore\ \psi_\mu(N)-a^{2l-2}b^{2m-2}\psi_\mu\left(\frac{N}{a^{l-1}b^{m-1}}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}$$

$$\times\left\{M(a^{2l+1}b^{2m+1}c^{3n}\dots)+\frac{\mu+3}{2}M(a^{2l+2}b^{2m+2}c^{4n}\dots)\right\},$$

and by proceeding in the same manner we finally arrive at

$$\psi_\mu(N)-a^{2l-2}b^{2m-2}c^{2n-2}\dots\psi_\mu\left(\frac{N}{a^{l-1}b^{m-1}c^{n-1}\dots}\right)$$

$$=-\frac{\mu(\mu+1)(\mu+2)}{24}\left\{M(a^{2l+1}b^{2m+1}c^{2n+1}\dots)+\frac{\mu+3}{2}M(a^{2l+2}b^{2m+2}c^{2n+2}\dots)\right\}$$

$$\dots\dots\dots\dots\dots(3).$$

If N_0 be written for the number $abc\,\ldots$, this last result is

$$\psi_\mu(N) = \frac{N^2}{N_0^2}\psi_\mu(N_0) - \frac{\mu(\mu+1)(\mu+2)}{24}\left\{M(N^2N_0) + \frac{\mu+3}{2}M(N^2N_0^2)\right\} \quad \ldots\ldots\ldots\ldots\ldots(4).$$

The determination of the divisibility of $\psi_\mu(N)$ by the prime factors of N is therefore reduced to the determination of the same thing with regard to $\psi_\mu(N_0)$.

4. Let p be a prime, and N any number prime to p. To obtain the numbers less than Np and prime to it we must deduct from the numbers less than Np and prime to N those which are multiples of p. But these last are just p multiplied by the numbers less than N and prime to N; accordingly $\psi_\mu(Np)$ is equal to

$$f_\mu(N, 0) + f_\mu(N, 1) + f_\mu(N, 2) + \ldots + f(N, p-1) - \frac{1}{p^\mu} f_\mu(N, 0).$$

Making use of equation (1) of § 2 to transform the expression on the right-hand side, by a method similar to that of § 3, we obtain

$$\begin{aligned}\psi_\mu(Np) &= p^2\psi_\mu(N) - \frac{\mu(\mu+1)(\mu+2)}{24}p^2(p^2-1) \\ &\qquad \times\left\{N^2\psi_{\mu+3}(N) + \frac{\mu+3}{2}M(N^4)\right\} - \frac{1}{p^\mu}f_\mu(N, 0) \\ &= \frac{p^{\mu+2}-1}{p^\mu}\psi_\mu(N) - \frac{\mu(\mu+1)(\mu+2)}{24}p^2(p^2-1) \\ &\qquad \times\left\{N^2\psi_{\mu+3}(N) + \frac{\mu+3}{2}M(N^4)\right\} \quad \ldots\ldots\ldots\ldots\ldots (5).\end{aligned}$$

By means of this formula we can reduce the determination of the divisibility of $\psi_\mu(abc\,\ldots\,k)$ to the determination of the same thing with regard to $\psi_\mu(bc\,\ldots\,k)$, and so, by successive applications of the formula, to the determination of it with regard to $\psi_\mu(k)$, which question has been solved in § 1. By means therefore of (3) or (4) and (5), it can be found how many times each of the prime factors of N divides $\psi_\mu(N)$. It will be seen presently that $\psi_{\mu+3}(N)$, where $\mu+3$ is an even number, is in general a multiple of N, so that in (5) we may often write $M(N)$ for $\psi_{\mu+3}(N)$; but there are several exceptions to this (see below, § 6). If N is a prime number it has been seen in § 1 that $\psi_{\mu+3}(N)$ is a multiple

of N unless $\mu+3 = M(N-1)$; thus, if a and b are two primes,

$$\psi_\mu(ab) = \frac{b^{\mu+2}-1}{b^\mu}\psi_\mu(a) - \frac{\mu(\mu+1)(\mu+2)}{24} b^2(b^2-1)M(a^4)\ldots\ldots(6),$$

unless $\mu+3 = M(a-1)$, when $M(a^3)$ must be substituted in place of $M(a^4)$.

5. In any case, whether $\psi_{\mu+3}(N)$ is or is not divisible by N, we have always

$$\psi_\mu(Np) = \frac{p^{\mu+2}-1}{p^\mu}\psi_\mu(N) - \frac{\mu(\mu+1)(\mu+2)}{24} p^2(p^2-1)M(a^3),$$

so that, if $a, b, c \ldots k$ are prime numbers,

$$\psi_\mu(abc\ldots k) = \frac{a^{\mu+2}-1}{a^\mu}\psi_\mu(bc\ldots k) - \frac{\mu(\mu+1)(\mu+2)}{24} M(b^3c^3\ldots k^3)$$

$$= \frac{a^{\mu+2}-1}{a^\mu}\left\{\frac{b^{\mu+2}-1}{b^\mu}\psi_\mu(c\ldots k) - \frac{\mu(\mu+1)(\mu+2)}{24} M(c^3\ldots k^3)\right\}$$

$$- \frac{\mu(\mu+1)(\mu+2)}{24} M(b^3c^3\ldots k^3)$$

$$= \frac{(a^{\mu+2}-1)(b^{\mu+2}-1)}{a^\mu b^\mu}\psi_\mu(c\ldots k) - \frac{\mu(\mu+1)(\mu+2)}{24} M(c^3\ldots k^3)$$

$$= \&\text{c.}$$

$$= \frac{(a^{\mu+2}-1)(b^{\mu+2}-1)(c^{\mu+2}-1)\ldots}{a^\mu b^\mu c^\mu\ldots}\psi_\mu(k) - \frac{\mu(\mu+1)(\mu+2)}{24} M(k^3)$$

$$\ldots\ldots\ldots\ldots\ldots(7).$$

But by (3)

$$\psi_\mu(a^l b^m c^n\ldots) = a^{2l-2}b^{2m-2}c^{2n-2}\ldots\psi_\mu(abc\ldots k)$$

$$- \frac{\mu(\mu+1)(\mu+2)}{24} M(a^{2l+1}b^{2m+1}c^{n+1}\ldots) \ldots\ldots\ldots\ldots (8).$$

Combining this with (7), after interchanging in the latter formula a and k for convenience,

$$\psi_\mu(a^l b^m c^n\ldots) = b^{2m+\mu-2}c^{2n+\mu-2}\ldots(b^{\mu+2}-1)(c^{\mu+2}-1)\ldots a^{2l-2}\psi_\mu(a)$$

$$- \frac{\mu(\mu+1)(\mu+2)}{24} M(a^{2l+1}) \ldots\ldots\ldots\ldots (9),$$

where a is any one of the prime factors of N.

Having regard to what has been proved in § 1 as to $\psi_\mu(a)$, and remembering that $\mu(\mu+1)(\mu+2)$ is always divisible by 6, we deduce from (8) the following conclusions:—

If a is any prime except 2 or 3, $\psi_\mu(N)$ is in general divisible by a^{2l}; but if $\mu+1 = M(a-1)$, then only by a^{2l-1}, unless at the same time $\mu = M(a)$, when $\psi_\mu(N)$ is divisible by a^{2l} as in general.

If a is 3, $\psi_\mu(N)$ is divisible by a^{2l} when μ is an odd multiple of 3, but in all other cases only by a^{2l-1}.

If a is 2, $\psi_\mu(N)$ is divisible by a^{2l-1}, except when the number of prime factors of N is less than 2 (*i.e.*, when $N = a^l$ simply), in which case $\psi_\mu(N)$ is divisible by a^{2l-2} only. This follows since $\mu+2$ is odd, and therefore $b^{\mu+2}-1$, $c^{\mu+2}-1$, &c. are each divisible by 2.

Should any of the expressions $b^{\mu+2}-1$, $c^{\mu+2}-1$, &c., or $\mu(\mu+1)(\mu+2)$ be a multiple of a, or of a power of a, then $\psi_\mu(N)$ may be divisible by a higher power of a than that given above; and this will frequently be the case.

6. By proceeding exactly as in § 1, it is seen that

$$\psi_\mu(N) = \tfrac{1}{2}M(N^2) - \frac{\mu}{2} N\psi_{\mu+1}(N).$$

Thus the divisibility of $\psi_{\mu+1}(N)$, where $\mu+1$ is *even*, by any prime factor a of N, depends on the divisibility of $\psi_\mu(N)$ by a, and this has been determined in § 5.

In particular, let $\mu = 3$; then

$$\psi_4(N) = \tfrac{1}{3}M(N) - \frac{2}{3N}\psi_3(N).$$

Now by § 5, $\psi_3(N)$ will be divisible in general by a^{2l}, and this will be so even if $a = 3$; but if $a = 5$ it is divisible by a^{2l-1} only, and if $a = 2$ by a^{2l-1} in all cases except where N is equal merely to a^l, when $\psi_3(N)$ will be divisible only by a^{2l-2}.

Accordingly, $\psi_4(N)$ is in general divisible by a^l, the exceptions being when a is 2, 3, or 5.

If $a = 5$, $\psi_4(N)$ is divisible by a^{l-1}.

If $a = 3$, $\psi_4(N)$ is divisible by a^{l-1}.

If $a = 2$, $\psi_4(N)$ follows the rule of being divisible by a^l unless $N = a^l$ merely, in which case $\psi_4(N)$ is divisible only by a^{l-1}.

The conclusion is therefore that $\psi_4(N) = \dfrac{1}{3.5}M(N)$ in every case except where N is simply a power of 2, when $\psi_4(N) = \tfrac{1}{3}M(N)$.

7. I proceed to consider in somewhat greater detail the case of $\mu = 1$, *i.e.*, that of the sum of the reciprocals of the numbers less than a number N and prime to it. When N is a prime number, p suppose, the result corresponding to that of § 1 may be proved conveniently as follows:—

The congruence

$$\{x^2+1(p-1)\}\{x^2+2(p-2)\}\dots\left\{x^2+\frac{p-1}{2}\,\frac{p+1}{2}\right\}-(x^{p-1}-1)\equiv 0 \pmod{p},$$

is satisfied by $p-1$ values of x, viz., $1, 2, 3 \dots p-1$. For $x^2+r(p-r)$ becomes pr when $x=r$ and $p(p-r)$ when $x=p-r$, so that it is divisible by p in either case; and $x^{p-1}-1$ is divisible by p when x has any one of the stated values, by Fermat's theorem. Now, since the congruence is only of degree $p-3$, it must be identical; therefore the numbers $1(p-1)$, $2(p-2)$, $3(p-3)$, ... $\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)$ are such that their sum, product two and two, ... $\frac{p-3}{2}$ and $\frac{p-3}{2}$ together, are all divisible by p. The last of these gives

$$\underline{|p-1}\left\{\frac{1}{1(p-1)}+\frac{1}{2(p-2)}+\dots+\frac{1}{\left(\frac{p+1}{2}\right)\left(\frac{p+1}{2}\right)}\right\},$$

or
$$\frac{1}{p}\,\underline{|p-1}\left\{\frac{1}{1}+\frac{1}{p-1}+\frac{1}{2}+\frac{1}{p-2}+\dots+\frac{1}{\frac{p-1}{2}}+\frac{1}{\frac{p+1}{2}}\right\},$$

divisible by p; so that

$$\underline{|p-1}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p-1}\right)$$

is divisible by p^2.

Since the congruence is of degree $p-3$, p must be greater than 3, and the argument does not hold for $p=2$ or $p=3$. In fact, if $p=3$, $\psi_1(p)$ is divisible only by p; and if $p=2$, $\psi_1(p)$ reduces to 1 and is not divisible by p at all.

8. For the case of $\mu = 1$ the formula (5) becomes

$$\psi_1(Np)=\frac{p^3-1}{p}\psi_1(N)-\tfrac{1}{4}p^2(p^2-1)\{N^3\psi_4(N)+2M(N^4)\}\quad\dots(10),$$

which in accordance with what has been proved in § 6 may be written,

$$\psi_1(Np)=\frac{p^3-1}{p}\psi_1(N)-\frac{p^2(p^2-1)}{3.4.5}M(N^4)\quad\dots\dots\dots(11),$$

unless N is simply a power of 2, say 2^n, in which case

$$\psi_1(Np) = \frac{p^3-1}{p}\psi_1(N) - \frac{p^2(p^2-1)}{8} M(N^4),$$

or
$$\psi_1(2^n p) = \frac{p^3-1}{p}\psi_1(2^n) - p^2(p^2-1) M(2^{4n-3}) \quad \ldots\ldots\ldots(12).$$

9. Writing in (9), $N = a$, $p = b$, where a and b are primes,

$$\psi_1(ab) = \frac{b^3-1}{b}\psi_1(a) - \frac{b^2(b^2-1)}{60} M(a^4).$$

Again, writing $N = ab$, $p = c$, where a, b, c are primes,

$$\psi_1(abc) = \frac{c^3-1}{c}\psi_1(ab) - \frac{c^2(c^2-1)}{60} M(a^4 b^4)$$

$$= \frac{(b^3-1)(c^3-1)}{bc}\psi_1(a) - \frac{b^2(b^2-1)(c^3-1)}{60c} M(a^4) - \frac{b^4 c^2(c^2-1)}{60} M(a^4)$$

$$= \frac{(b^3-1)(c^3-1)}{bc}\psi_1(a) - \frac{b^2(c-1)}{60c} M(a^4).$$

Then, writing $N = abc$, $p = d$, where a, b, c, d are primes,

$$\psi_1(abcd) = \frac{(b^3-1)(c^3-1)(d^3-1)}{bcd}\psi_1(a) - \frac{d-1}{60cd} M(a^4),$$

and by proceeding in the same manner we obtain

$$\psi_1(abcd\ldots k) = \frac{(b^3-1)(c^3-1)(d^3-1)\ldots(k^3-1)}{bcd\ldots k}\psi_1(a) + \frac{k-1}{60cd\ldots k} M(a^4),$$

where k stands for any of the prime factors of $N = abcd\ldots k$. From this formula conclusions may be drawn with regard to $\psi_1(N)$ in the same manner as has been done in § 5 with regard to $\psi_\mu(N)$. It may be noticed that the numerator of $\psi_1(N)$ is divisible by $k-1$, and so by each of the numbers $a-1$, $b-1$, &c.; but the denominator of $\psi_1(N)$ will not in general be prime to these numbers. In working out any numerical case in practice the best method will be to make use of (8) and of successive applications of (9) or (10).

10. Formulæ similar in character to the foregoing may in the same way be proved for the sum [say $\phi_\mu(N)$] of the μ^{th} powers of the numbers less than N and prime to it, μ being a positive odd integer. For the reasons explained at the beginning of the paper it may be sufficient to give a statement of the results for this case, merely working out the formula corresponding to (1).

If $F_\mu(N, n)$ denote the sum of the μ^{th} powers of the numbers prime to N which lie between nN and $(n+1)N$, then exactly as in § 2, using a similar notation,

$$\frac{1}{2n+1}F_\mu(N,n)-F_\mu(N,0)$$

$$=\tfrac{1}{2}\Sigma\left\{\frac{(nN+x)^\mu+(nN+N-x)^\mu}{2n+1}-\frac{x^\mu+(N-x)^\mu}{1}\right\}$$

$$=\tfrac{1}{2}\Sigma x^\mu\left\{\frac{\left(1+\frac{nN}{x}\right)^\mu+(-1)^\mu\left(1+\frac{nN+N}{x}\right)^\mu}{2n+1}-1+(-1)^\mu\left(1-\frac{N}{x}\right)^\mu\right\}$$

$$=\frac{N^3}{2}(n^2+n)\frac{\mu(\mu-1)(\mu-2)}{6}\left\{\phi_{\mu-3}(N)-\frac{\mu-3}{2}\phi_{\mu-4}(N)+\dots\right\},$$

the coefficient of the general term $\phi_{\mu-m-1}(N)$ being

$$\frac{N^3}{2}\,\frac{\mu(\mu-1)\dots(\mu-m)}{1.2\dots(m+1)}\left\{\frac{(n+1)^{m+1}-(-n)^{m+1}}{2n+1}-1\right\}.$$

This is the formula corresponding in this case to (1). Corresponding to the results of § 1, we have precisely similar ones with $\mu-1$ written in place of the $\mu+1$ which occurs there. Corresponding to (2), (3), (5), (7), (9), respectively, we find

$$\phi_\mu(N)=a^2\phi_\mu\left(\frac{N}{a}\right)$$

$$+\frac{\mu(\mu-1)(\mu-2)}{24}a^2(a^2-1)\left\{\frac{N^3}{a^3}\phi_{\mu-3}\left(\frac{N}{a}\right)-\frac{\mu+3}{2}M\left(\frac{N^4}{a^4}\right)\right\},$$

$$\phi_\mu(N)=a^{2l-2}b^{2m-2}c^{2n-2}\dots\phi_\mu\left(\frac{N}{a^{l-1}b^{m-1}c^{n-1}\dots}\right)$$

$$+\frac{\mu(\mu-1)(\mu-2)}{24}\left\{M(a^{2l+1}b^{2m+1}c^{2n+1}\dots)-\frac{\mu-3}{2}M(a^{2l+2}b^{2m+2}c^{2n+2}\dots)\right\},$$

$$\phi_\mu(Np)=-p^2(p^{\mu-2}-1)\phi_\mu(N)$$

$$+\frac{\mu(\mu-1)(\mu-2)}{24}p^2(p^2-1)\left\{N^3\phi_{\mu-3}(N)-\frac{\mu-3}{2}M(N^4)\right\},$$

$$\phi_\mu(abc\dots)=a^2b^2c^2\dots(1-a^{\mu-2})(1-b^{\mu-2})(1-c^{\mu-2})\dots\phi_\mu(k)$$

$$+\frac{\mu(\mu-1)(\mu-2)}{24}M(k^3),$$

and $\phi_\mu(a^l b^m c^n \ldots) = b^{2m} c^{2n} \ldots (1-a^{\mu-2})(1-b^{\mu-2})(1-c^{\mu-2}) \ldots a^{2l-2} \phi_\mu(a)$

$$+\frac{\mu(\mu-1)(\mu-2)}{24} M(a^{2l+1}).$$

If therefore $-\mu$ be written for μ in the conclusions as to $\psi_\mu(N)$ given in § 5, these will apply in the case of $\phi_\mu(N)$.

11. It is necessary in applying the foregoing formulæ to bear in mind their precise meaning, as otherwise wrong conclusions may easily be drawn from them. Such a formula as (11), for instance, shows that, if $N = 2^\alpha 3^\beta 4^\gamma a^l b^m c^n \ldots$, the numerator of the expression

$$\frac{p^3-1}{p} \psi_1(N) - \psi_1(Np)$$

is divisible by $2^{4\alpha-2} 3^{4\beta-1} 5^{4\gamma-1} a^{4l} b^{4m} c^{4n} \ldots$ in any case, and by any further power of any prime factor of N which may happen to be contained in p^3-1. The formula must not be interpreted as affirming anything concerning the divisibility of the numerator of the above-mentioned expression by p^3 or p^3-1, because p^3 and p^3-1 need not be prime to 3, 4, or 5, nor to the denominator of the expression denoted in (11) by $M(N^4)$. For example, if $N = 2^2$ and $p = 3$, formula (12) shows that

$$\begin{aligned} \frac{26}{3} \psi_1(4) - \psi_1(12) &= 9 \,.\, 8 M(2^5) \\ &= 9 \,.\, M(2^8), \end{aligned}$$

and the conclusion is that the expression on the left-hand side is divisible by 2^8, which is correct, since it is equal to

$$\begin{aligned} \frac{26}{3}\left(\frac{1}{1}+\frac{1}{3}\right) - \left(\frac{1}{1}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}\right) \\ = \frac{35072}{1155} \\ = \frac{2^8 \,.\, 137}{1155}. \end{aligned}$$

But it would be wrong to infer that the expression is divisible by 9, as in fact can be seen at once on going back to formula (10) from which (12) was deduced. For $\psi_4(N)$, which occurs on the right-hand side of (10), is in this case $\frac{1}{1} + \frac{1}{81}$ or $\frac{82}{81}$, and so involves a

power of 9 in its denominator which cancels against the 9 arising from the factor p^2-1.

12. I add a few numerical examples of the application of the formulæ.

Ex. 1. $N = 21$, $\mu = 1$.

By (11), $\psi_1(21) = \dfrac{342}{7}\psi_1(3) - \dfrac{49\,.\,48}{60}M(3^4) = M(3^3)$,

and $$\psi_1(21) = \frac{26}{3}\psi_1(7) - \frac{9\,.\,8}{60}M(7^4) = M(7^2),$$

therefore $$\psi_1(21) = M(3^3\,.\,7^2),$$

and then, by (8),

$$\begin{aligned}\psi_1(3^l\,.\,7^m) &= 3^{2l-2}\,.\,7^{2m-2}\,\psi_1(21) - \tfrac{1}{4}M(3^{2l+1}\,.\,7^{2m+1})\\ &= M(3^{2l+1}\,.\,7^{2m}).\end{aligned}$$

Ex. 2. $N = 505$, $\mu = 1$.

By (11),

$$\psi_1(505) = \frac{1030300}{101}\psi_1(5) - \frac{10201\,.\,10200}{60}M(5^4) = M(5^4),$$

and $$\psi_1(505) = \frac{124}{5}\psi_1(101) - \frac{24\,.\,25}{60}M(101^4) = M(101^2),$$

therefore $$\psi_1(505) = M(5^4\,.\,101^2).$$

Ex. 3. $N = 78$, $\mu = 1$.

Here $$\psi_1(78) = \frac{26}{3}\psi_1(26) - \frac{9\,.\,8}{60}M(26^4),$$

$$\psi_1(78) = \frac{2196}{13}\psi_1(6) - \frac{169\,.\,168}{60}M(6^4),$$

$$\psi_1(26) = \frac{7}{2}\psi_1(13) - \frac{4\,.\,3}{60}M(13^4),$$

$$\psi_1(6) = \frac{26}{3}\psi_1(2) - 9\,.\,8\,M(2), \text{ by (12)},$$

$$\psi_1(6) = \frac{7}{2}\psi_1(3) - \frac{4\,.\,3}{60}M(3^4),$$

by (11). Therefore

$$\psi_1(6) = M(2), \quad \psi_1(6) = M(3), \quad \psi_1(26) = M(13^2);$$

and $$\psi_1(78) = M(2^3 \,.\, 3^3 \,.\, 13^3) = M(78^3).$$

Ex. 4. $N = 28$, $\mu = 13$.

By (3),

$$\psi_{13}(28) = 2^2\psi_{13}(14) - \frac{13\,.\,14\,.\,15}{24}\{M(2^4\,.\,7^2) + 8M(2^5\,.\,7^3)\}$$

$$= 2^2\psi_{13}(14) - M(2^2\,.\,7^3),$$

and, by (6),

$$\psi_{13}(14) = \frac{2^{15}-1}{2^{13}}\psi_{13}(7) - \frac{13\,.\,14\,.\,15}{24}4\,.\,3\,M(7^4) = M(7^3),$$

$$\psi_{13}(14) = \frac{7^{15}-1}{7^{13}}\psi_{13}(2) - \frac{13\,.\,14\,.\,15}{24}\,.\,49\,.\,48\,M(2^3) = M(2),$$

therefore $$\psi_{13}(28) = M(2^3\,.\,7^3) - M(2^2\,.\,7^3)$$

$$= M(2^2\,.\,7^3).$$

Ex. 5. $N = 56$, $\mu = 11$.

By (3),

$$\psi_{11}(56) = 2^4\psi_{11}(14) - \frac{11\,.\,12\,.\,13}{24}\{M(2^7\,.\,7^3) + 7M(2^8\,.\,7^4)\},$$

and, by (6),

$$\psi_{11}(14) = \frac{2^{13}-1}{2^{11}}\psi_{11}(7) - \frac{11\,.\,12\,.\,13}{24}4\,.\,3M(7^4),$$

$$\psi_{11}(14) = \frac{7^{13}-1}{7^{11}}\psi_{11}(2) - \frac{11\,.\,12\,.\,13}{24}49\,.\,48\,M(2^3).$$

Now $\psi_{11}(7)$ is divisible by 7 (not by 7^2), since $11+1 = M(6)$; and $2^{13}-1 = 8191$, which is a prime number; therefore $\psi_{11}(14) = M(7)$. Again, $7^{13}-1$ is divisible by 2 (not by 2^2); $\therefore\ \psi_{11}(14) = M(2)$.

Therefore $$\psi_{11}(56) = M(2^5\,.\,7) - \tfrac{1}{2}M(2^7\,.\,7^3)$$

$$= M(2^5\,.\,7).$$

Lamé's Differential Equation. *By* A. G. GREENHILL.

[*Read May 10th*, 1888.]

1. This differential equation, in the form

$$\frac{1}{y}\frac{d^2y}{dx^2} = n(n+1)\,k^2\,\mathrm{sn}^2x + h,$$

employing Jacobi's notation of the elliptic functions, has been solved completely by M. Hermite (*Sur quelques applications des fonctions elliptiques*, Paris, 1885, a collection of articles from the *Comptes Rendus*).

But the advantage of the notation of Weierstrass has been pointed out by M. Halphen (*Mémoire sur la réduction des équations différentielles linéaires aux formes intégrables*, Paris, 1884); the differential equation of Lamé then takes the form

$$\frac{1}{y}\frac{d^2y}{dx^2} = n(n+1)\,\wp x + h \quad\text{..................... (1)},$$

where $\wp x$ is Weierstrass's elliptic function; and the solution of Hermite then takes the form

$$y = CF(x) + C'F(-x) \quad\text{......................(2)},$$

where $F(x)$ is a *doubly periodic function of the second kind* (*fonction doublement périodique de seconde espèce*), which, according to Hermite, can be expressed in the form

$$F(x) = D_x^{n-1}\,\phi(x) - A_1 D_x^{n-3}\,\phi(x) + A_2 D_x^{n-5}\,\phi(x) - \dots \quad\text{......(3)},$$

where $\phi(x)$, called the simple element, expressed by Weierstrass's σ and ζ functions, is of the form

$$\phi(x) = \frac{\sigma(x+\omega)}{\sigma x\,\sigma\omega}\exp(\lambda - \zeta\omega)\,x \quad\text{................ (4)};$$

and Halphen has shown (*Fonctions elliptiques*) that, when $\lambda = 0$, $\phi(x)$ satisfies the differential equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = \frac{\phi''x}{\phi x} = 2\wp x + \wp\omega \quad\text{......................... (5)},$$

Lamé's differential equation for $n = 1$.

2. In order to obtain the coefficients A_1, A_2, ... in (3), the functions $F(x)$ and $\wp x$ are expanded in powers of x in the neighbourhood of $x = 0$, in the form

$$(-1)^{n-1} F(x) = \frac{(n-1)!}{x^n} - \frac{A_1 (n-3)!}{x^{n-2}} + \frac{A_2 (n-5)!}{x^{n-4}} - \ldots\ldots \quad (6),$$

$$\wp x = \frac{1}{x^2} + \frac{g_2 x^2}{20} + \frac{g_3 x^4}{28} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\phi(x) = \frac{1}{x} + \lambda + (\lambda^2 + P_2)\frac{x}{2!} + (\lambda^3 + 3P_2\lambda + P_3)\frac{x^2}{3!} + \ldots$$

(Halphen, *Fonctions elliptiques*, I., p. 231), and then, substituting in the differential equation, we find

$$A_1 = \frac{(n-1)(n-2)}{2(2n-1)} h,$$

$$A_2 = \frac{(n-1)(n-2)(n-3)(n-4)}{8(2n-1)(2n-3)} \left\{ h^2 - \frac{n(n+1)(2n-1)}{10} g_2 \right\},$$

$$A_3 = \ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

3. But, if we suppose that a particular solution $\Phi(x)$ of (1) is of the form

$$\Phi(x) = \frac{\sigma(x+a_1)\,\sigma(x+a_2)\ldots\sigma(x+a_n)}{\sigma a_1\,\sigma a_2 \ldots \sigma a_n\,(\sigma x)^n} \exp(-\zeta a_1 - \zeta a_2 - \ldots - \zeta a_n)\,x,$$

the product of n simple elements, each of the form

$$\phi(x) = \frac{\sigma(x+a)}{\sigma a\,\sigma x} \exp(-\zeta a)\,x,$$

and if we seek to satisfy the differential equation (1), we shall have, putting $y = \Phi(x)$,

$$\frac{1}{y}\frac{dy}{dx} = \Sigma\{\zeta(x+a) - \zeta x - \zeta a\}$$

$$= \Sigma \tfrac{1}{2} \frac{\wp' x - \wp' a}{\wp x - \wp a},$$

$$\frac{1}{y}\frac{d^2 y}{dx^2} = \Sigma\{\wp x - \wp(x+a)\} + \left(\frac{1}{y}\frac{dy}{dx}\right)^2$$

$$= \Sigma\{\wp x - \wp(x+a)\} + \tfrac{1}{4}\Sigma\left(\frac{\wp' x - \wp' a}{\wp x - \wp a}\right)^2$$

$$+ \tfrac{1}{2}\Sigma\left(\frac{\wp' x - \wp' a_r}{\wp x - \wp a_r}\right)\left(\frac{\wp' x - \wp' a_s}{\wp x - \wp a_s}\right)$$

$$= 2n\wp x + \Sigma\wp a + \tfrac{1}{2}\Sigma(\ )(\ )$$

$$= n(n+1)\wp x + (2n-1)\Sigma\wp a,$$

provided that we can make

$$\tfrac{1}{2}\Sigma\left(\frac{\wp' x-\wp' a_r}{\wp x-\wp a_r}\right)\left(\frac{\wp' x-\wp' a_s}{\wp x-\wp a_s}\right)=n\,(n-1)\,\wp x+(2n-2)\,\Sigma\wp a,$$

and then $$h=(2n-1)\,\Sigma\wp a.$$

The necessary conditions for the above relation to hold are

$$\Sigma\wp' a=0,\quad \Sigma\wp a\,\wp' a=0,\quad \Sigma\,(\wp a)^2\,\wp' a=0,\ \ldots\ \Sigma\,(\wp a)^{n-2}\,\wp' a=0,$$

as Brioschi has demonstrated (*Comptes Rendus*, XCII., p. 325); then

$$\Sigma\,(\wp a)^{n-1}\,\wp' a=C=f'(\rho)\,\wp' v,$$

putting, with Brioschi, $h=n\,(2n-1)\,\rho$, and then $\rho=\wp v$.

Here $f(\wp x)$ denotes the product $\Phi(x)\,\Phi(-x)$ of two particular solutions $\Phi(x)$ and $\Phi(-x)$ of (1); and thus

$$f(\wp v)=\Pi\,(\wp v-\wp a),$$

and
$$\Sigma\wp a=n\rho=n\wp v,$$
$$h=n\,(2n-1)\,\wp v.$$

These conditions of Brioschi may be replaced by

$$\Sigma\wp' a=0,\quad \Sigma\wp''' a=0,\ \ldots\ \Sigma\wp^{(2n-3)}\,a=0,$$

and
$$\Sigma\wp^{(2n-1)}\,a=Nf'(\rho)\,\wp' v.$$

4. In order to compare the two solutions $F(x)$ and $\Phi(x)$ of the differential equation (1), it will be necessary to decompose $\Phi(x)$ into simple elements of the form $\phi(x)$ of (4), and of its derivatives (Halphen, *Fonctions elliptiques*, I., p. 228), and then we shall find

$$a_1+a_2+a_3+\ldots+a_n=\omega,$$

$$\zeta\omega-\zeta a_1-\zeta a_2-\ldots-\zeta a_n=\lambda.$$

Differentiating $\Phi(x)$ logarithmically,

$$\frac{\Phi'(x)}{\Phi(x)}=\Sigma\left\{\zeta\,(x+a)-\zeta x-\zeta a\right\}$$

$$=-\frac{n}{x}+B_2\frac{x}{1\,!}+B_3\frac{x^2}{2\,!}+B_4\frac{x^3}{3\,!}+\ldots$$

where
$$B_2 = -\Sigma \wp a = -n\wp v,$$
$$B_3 = -\Sigma \wp' a = 0,$$
$$B_4 = -\Sigma \wp'' a + \frac{ng_2}{10}, \quad B_5 = 0, \ldots$$

and we have
$$\Phi(x) = \frac{1}{x^n}\left(1 + P_2\frac{x^2}{2!} + P_4\frac{x^4}{4!} + \ldots\right),$$

where
$$P_2 = B_2, \quad P_4 = B_4 + 3B_2^2,$$
$$P_6 = B_6 + 15B_2B_4 + 15B_2^3, \ldots.$$

Consequently the decomposition of $\Phi(x)$ into simple elements is of the form
$$(-1)^{n-1}\Phi(x) = \frac{D_x^{n-1}\phi(x)}{(n-1)!} + \frac{P_2}{2!}\frac{D_x^{n-3}\phi(x)}{(n-3)!} + \frac{P_4}{4!}\frac{D_x^{n-5}\phi(x)}{(n-5)!} + \ldots,$$

and
$$F(x) = (n-1)!\,(-1)^{n-1}\Phi(x),$$
$$P_2 = -n\wp v,$$
$$P_4 = \frac{3n}{2n-3}\left\{n(2n-1)\wp^2 v - \frac{1}{10}(n+1)g_2\right\}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

whence
$$\Sigma \wp a = n\wp v,$$
$$\Sigma \wp'' a = -\frac{n^2\wp'' v}{2n-3},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$\Sigma \wp^{(2n)} a$ being, in general, an integral function of $\wp v$.

5. The differential equation for $Y = f(\wp x)$, the product of $\Phi(x)$ and $\Phi(-x)$, two solutions of Lamé's equation, is easily formed (Halphen, *Fonctions elliptiques*, II., p. 498).

For, if we take the linear differential equation of the second order in its canonical form,
$$\frac{1}{y}\frac{d^2y}{dx^2} = I,$$

and if y and z are two particular solutions, so that
$$Y = yz,$$

then, denoting differentiation by accents,
$$Y' = y'z + yz',$$
$$Y'' = y''z + 2y'z' + yz''$$
$$= 2Iyz + 2y'z',$$

or
$$Y''-2IY = 2y'z',$$

and
$$Y'''-2IY'-2I'Y = 2y''z'+2y'z''$$
$$= 2I(yz'+y'z) = 2IY',$$

or
$$Y'''-4IY'-2I'Y = 0,$$

a differential equation of the third order for Y, the general solution of which is
$$ay^2+2byz+cz^2,$$

with y^2, yz, and z^2 for particular solutions.

6. In Lamé's differential equation
$$I = n(n+1)\wp x+h,$$
$$Y'''-4\{n(n+1)\wp x+h\}Y'-2n(n+1)\wp' xY = 0,$$

and this equation has, as a particular solution, a rational integral function of $\wp x$ of the n^{th} degree, which is
$$Y = f(\wp x) = \Pi(\wp x-\wp a).$$

Then
$$\frac{Y'}{Y} = \frac{y'}{y}+\frac{z'}{z} = \Sigma\frac{\wp' x}{\wp x-\wp a};$$

also, since
$$y''z-yz'' = 0,$$

therefore
$$y'z-yz' = C,$$

or
$$\frac{y'}{y}-\frac{z'}{z} = \frac{C}{Y} = \frac{C}{\Pi(\wp x-\wp a)},$$

and this, according to Brioschi, may, when resolved into partial fractions, be replaced by
$$\Sigma\frac{\wp' a}{\wp x-\wp a}.$$

Thus
$$\frac{y'}{y} = \Sigma\tfrac{1}{2}\frac{\wp' x+\wp' a}{\wp x-\wp a},$$
$$\frac{z'}{z} = \Sigma\tfrac{1}{2}\frac{\wp' x-\wp' a}{\wp x-\wp a},$$

leading to the solution given above in § 3.

7. For certain particular values of h we obtain the solutions originally considered by Lamé (Ferrers, *Spherical Harmonics*, Chap. VI.), which

are rational integral functions of $\wp x$ and $\wp' x$; but these have recently been shown by Halphen (*Fonctions elliptiques*, II., p. 273) to be identical with binary forms Z, which are identical with the covariant

$$\Phi_{22} Z_{11} - 2\Phi_{12} Z_{12} + \Phi_{11} Z_{22},$$

composed of Z and a form Φ of the fourth degree.

8. Consider the particular cases of $n = 1$, 2, and 3.

Case I., $n = 1$. The differential equation is then

$$\frac{1}{y} \frac{d^2 y}{dx^2} = 2\wp x + \wp v,$$

the solution of which is

$$y = CF(x) + C'F(-x),$$

where
$$F(x) = \frac{\sigma(x+v)}{\sigma x \sigma v} \exp(-x\zeta v)$$

(Halphen, p. 235).

Case II., $n = 2$. The differential equation is then

$$\frac{1}{y} \frac{d^2 y}{dx^2} = 6\wp x + 6\wp v;$$

and then

$$F(x) = D_x \frac{\sigma(x+\omega)}{\sigma x \sigma \omega} \exp(\lambda - \zeta\omega) x,$$

$$= -\frac{\sigma(x+a_1)\,\sigma(x+a_2)}{\sigma a_1\, \sigma a_2\, (\sigma x)^2} \exp(-\zeta a_1 - \zeta a_2) x,$$

where
$$a_1 + a_2 = \omega,$$

$$\lambda = \zeta\omega - \zeta a_1 - \zeta a_2,$$

$$\wp a_1 + \wp a_2 = 2\wp v,$$

$$\wp' a_1 + \wp' a_2 = 0,$$

$$\wp v - \wp\omega = \frac{\wp'^2 v}{2\wp'' v},$$

$$\lambda = \tfrac{1}{2} \frac{\wp'\omega}{\wp v - \wp\omega} = \zeta\omega - \tfrac{1}{2}\zeta(\omega + v) - \tfrac{1}{2}\zeta(\omega - v) = \frac{\wp'\omega\, \wp'' v}{\wp'^2 v},$$

$$\lambda^2 - 3\wp\omega = 2(\wp v - \wp\omega) = \frac{\wp'\omega}{\lambda}, \text{ \&c.}$$

Case III., $n = 3$. The differential equation is

$$\frac{1}{y} \frac{d^2 y}{dx^2} = 12\wp x + 15\wp v,$$

and then
$$F(x) = D_x^2 \frac{\sigma(x+\omega)}{\sigma x \sigma \omega} \exp(\lambda - \zeta\omega) x$$
$$-3\wp v \frac{\sigma(x+\omega)}{\sigma x \sigma \omega} \exp(\lambda - \zeta\omega) x,$$
$$= 2 \frac{\sigma(x+a_1)\sigma(x+a_2)\sigma(x+a_3)}{\sigma a_1 \sigma a_2 \sigma a_3 (\sigma x)^3} \exp(-\zeta a_1 - \zeta a_2 - \zeta a_3) x,$$
where
$$a_1 + a_2 + a_3 = \omega,$$
$$\lambda = \zeta\omega - \zeta a_1 - \zeta a_2 - \zeta a_3,$$
$$\wp a_1 + \wp a_2 + \wp a_3 = 3\wp v,$$
$$\wp' a_1 + \wp' a_2 + \wp' a_3 = 0,$$
$$\Sigma \wp a \wp' a = 0,$$
$$\frac{\wp'\omega}{\lambda} = \lambda^2 - 3\wp\omega - 9\wp v$$
$$= 2(\wp v - \wp\omega) - \frac{3\wp'^2 v}{2\wp'' v}, \text{ \&c.}$$

In interpreting the results of M. Hermite (*Sur quelques applications*, &c., pp. 124–129) in this notation, we must take his
$$\Omega = \wp\omega, \quad \Omega_1 = \tfrac{1}{2}\wp'\omega,$$
$$\Omega_2 = \wp^2\omega - \tfrac{1}{5}g_2, \quad \Omega_3 = \tfrac{1}{2}\wp\omega\,\wp'\omega, \ldots$$
$$h = -5l = 15\wp v,$$
and generally
$$h = n(2n-1)\wp v,$$
$$a = \tfrac{3}{4}g_2, \quad b = \tfrac{27}{4}g_3, \quad 4a^3 - b^2 = \tfrac{27}{16}\Delta, \ldots .$$

The cases of $n = 4$ and $n = 5$ are also investigated by Halphen in his *Fonctions elliptiques*, II., p. 529, but the complexity increases very rapidly.

9. The origin of Lamé's differential equation in connection with physical problems relating to confocal quadric surfaces was explained in *Proc. Lond. Math. Soc.*, XVIII., p. 275, employing the notation of Weierstrass.

Putting, in the usual notation,
$$a^2+\lambda = \wp u - e_1, \quad b^2+\lambda = \wp u - e_2, \quad c^2+\lambda = \wp u - e_3,$$
$$a^2+\mu = \wp v - e_1, \quad b^2+\mu = \wp v - e_2, \quad c^2+\mu = \wp v - e_3,$$
$$a^2+\nu = \wp w - e_1, \quad b^2+\nu = \wp w - e_2, \quad c^2+\nu = \wp w - e_3,$$

then Poisson's equation becomes

$$(\wp v - \wp w)\frac{\partial^2\phi}{\partial u^2} + (\wp w - \wp u)\frac{\partial^2\phi}{\partial v^2} + (\wp u - \wp v)\frac{\partial^2\phi}{\partial w^2} = 0,$$

and supposing that ϕ may be decomposed into terms of the form UVW, where U is a function of u, V of v, and W of w only, then

$$(\wp v - \wp w)\frac{d^2 U}{U\,du^2} + (\wp w - \wp u)\frac{d^2 V}{V\,dv^2} + (\wp u - \wp v)\frac{d^2 W}{W\,dw^2} = 0,$$

equivalent to

$$\frac{1}{U}\frac{d^2 U}{du^2} = g\wp u + h,$$

$$\frac{1}{V}\frac{d^2 V}{dv^2} = g\wp v + h,$$

$$\frac{1}{W}\frac{d^2 W}{dw^2} = g\wp w + h;$$

and g must be put equal to $n(n+1)$ for the solution of these equations to be a *uniform* function.

It is usual to take $e_1 > e_2 > e_3$, so that we must suppose a^2, b^2, c^2 to be in ascending order of magnitude.

10. In dealing with spheroidal harmonics, two of these three quantities are equal.

For oblate spheroids, $b^2 = c^2$, and $e_2 = e_3$; and we can choose the constants so that

$$\wp u - e_1 = \cot^2 u, \quad \wp u - e_3 = \operatorname{cosec}^2 u,$$

by making

$$e_1 = \tfrac{2}{3}, \quad e_2 = e_3 = -\tfrac{1}{3}.$$

For prolate spheroids, $a^2 = b^2$, and $e_1 = e_2$, and then, by making

$$e_1 = e_2 = \tfrac{1}{3}, \quad e_3 = -\tfrac{2}{3},$$

$$\wp u - e_3 = \coth^2 u, \quad \wp u - e_1 = \operatorname{cosech}^2 u.$$

The corresponding Lamé equations are then of the form

$$\frac{1}{y}\frac{d^2 y}{du^2} = n(n+1)\operatorname{cosec}^2 u + h,$$

or

$$= n(n+1)\operatorname{cosech}^2 u + h,$$

the solution of which can be expressed in Hermite's manner by the corresponding degenerate circular or hyperbolic functions.

11. For instance, the solution in general being written

$$y = CF(x) + C'F(-x),$$

for the particular case of $(n = 1)$

$$\frac{1}{y}\frac{d^2y}{dx^2} = 2\operatorname{cosec}^2 x + \cot^2 a$$

we have $$F(x) = \frac{\sin(x+a)}{\sin x \sin a} e^{-x\cot a} = (\cot x + \cot a)\, e^{-x\cot a};$$

and for $(n = 2)$ $$\frac{1}{y}\frac{d^2y}{dx^2} = 6\operatorname{cosec}^2 x + \cot^2 a,$$

$$F(x) = \frac{d}{dx}\left\{\frac{\sin(x+b)}{\sin x \sin b} e^{-x\cot a}\right\},$$

where $$\cot b = \tfrac{1}{3}\cot a - \tfrac{2}{3}\tan a;$$

with corresponding expressions when the circular functions are replaced by hyperbolic functions; and so on for other particular cases which can be indefinitely multiplied.

12. A still more degenerate case is obtained by supposing that

$$e_1 = e_2 = e_3 = 0;$$

then $$a^2 = b^2 = c^2,$$

and $$\wp u = \frac{1}{u^2},$$

and we obtain the ordinary spherical harmonics as the solution of Laplace's equation.

Then Lamé's equation degenerates into

$$\frac{1}{y}\frac{d^2y}{dx^2} = \frac{n(n+1)}{x^2} + h,$$

the differential equation discussed in Boole's *Differential Equations*, p. 424; Forsyth's *Differential Equations*, p. 176; also by Glaisher.

Thus, if we take $n = 1$ and $h = q^2$, we have

$$y = CF(x) + C'F(-x),$$

the solution of $$\frac{1}{y}\frac{d^2y}{dx^2} = \frac{2}{x^2} + q^2,$$

where $$F(x) = \left(\frac{1}{x} + q\right) e^{-qx};$$

while $y = C\left(\frac{1}{x}\cos qx + q\sin qx\right) + C'\left(\frac{1}{x}\sin qx - q\cos qx\right)$

is the solution of $$\frac{1}{y}\frac{d^2y}{dx^2} = \frac{2}{x^2} - q^2.$$

13. The differential equation for the propagation of an impulsive jerk T along a uniform chain lying in a curve on a smooth table is

$$\frac{1}{T}\frac{d^2T}{ds^2} = \frac{1}{\rho^2},$$

and is therefore soluble in the manner explained above for curves in which the intrinsic equation

$$\frac{1}{\rho^2} = I = n(n+1)\wp s + h;$$

but these curves do not appear to possess any simple properties.

14. Consider the differential equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = \tfrac{1}{4}\operatorname{sech}^2 x;$$

this is the form assumed by the differential equation for K and K', given in Cayley's *Elliptic Functions*, p. 51, when we put

$$k^2 = \frac{1}{1+e^{2x}}, \quad k'^2 = \frac{1}{1+e^{-2x}};$$

or $$k^2 = \tfrac{1}{2}(1-\tanh x), \quad k'^2 = \tfrac{1}{2}(1+\tanh x),$$

so that its solution is $$y = CK + C'K';$$

or $$T = CK + C'K'$$

is the solution of $$\frac{1}{T}\frac{d^2T}{ds^2} = \frac{1}{\rho^2},$$

if $$\frac{1}{\rho^2} = \frac{1}{4c^2}\operatorname{sech}^2\frac{s}{c},$$

and then $$k^2 = \frac{1}{1+e^{2s/c}}, \quad k'^2 = \frac{1}{1+e^{-2s/c}},$$

or $$\cos 2\theta = k'^2 - k^2 = \tanh s/c,$$

θ denoting the modular angle, so that

$$\tfrac{1}{2}\pi - 2\theta = \operatorname{gd} s/c.$$

15. In this case the equation of the curve in which the chain lies may be evaluated; for

$$\frac{1}{\rho} = \frac{d\psi}{ds} = -\frac{\operatorname{sech} s/c}{2c},$$

taking the negative sign; and then

$$2\psi = \sin^{-1} \operatorname{sech} s/c,$$

$$\sin 2\psi = \operatorname{sech} s/c,$$

$$\cos 2\psi = \tanh s/c,$$

so that we find $$\theta = \psi.$$

Then $$\frac{dx}{ds} = \cos\psi = \sqrt{\{\tfrac{1}{2}(1+\cos 2\psi)\}}$$

$$= \sqrt{\{\tfrac{1}{2}(1+\tanh s/c)\}} = \frac{e^{s/c}}{\sqrt{(e^{2s/c}+1)}},$$

$$\frac{dy}{ds} = \sin\psi = \frac{e^{-s/c}}{\sqrt{(1+e^{-2s/c})}};$$

and integrating, from $s = 0$,

$$x/c = \sinh^{-1} e^{s/c} - \sinh^{-1} 1,$$

$$y/c = \sinh^{-1} 1 - \sinh^{-1} e^{-s/c};$$

or, putting $\sinh^{-1} 1 = a = \cosh^{-1}\sqrt{2} = \log(\sqrt{2}+1)$,

$$e^{s/c} = \sinh(x/c+a),$$

$$e^{-s/c} = \sinh(a-y/c),$$

so that $$\sinh(x/c+a)\sinh(a-y/c) = 1,$$

the Cartesian equation of the curve of the chain, a catenary in which the linear density varies as $e^{-s/c}$.

16. We see that

$$x/c+a = 0 \quad \text{and} \quad a-y/c = 0$$

are asymptotes; and, changing to them for coordinate axes,

$$\sinh x/c \sinh y/c = 1.$$

This may be written, $\sinh y/c = \operatorname{cosech} x/c$,

$$\cosh y/c = \coth x/c,$$

$$e^{y/c} = \coth x/c + \operatorname{cosech} x/c$$

$$= \coth \tfrac{1}{2}x/c,$$

or $$y/c = \log \coth \tfrac{1}{2} x/c,$$

or $$x/c = \log \coth \tfrac{1}{2} y/c.$$

17. Lamé's equation has received considerable attention of recent years, and has led to the discovery of a large class of differential equations, also soluble by elliptic functions, for which Halphen's Chapter XIII., t. II., *Fonctions elliptiques*, may be consulted.

Besides the references already given, the following articles may be consulted :—Brioschi, *Annali di Matematica*, IX., p. 11 ; Fuchs, *Annali di Matematica*, IX., p. 25 ; Brioschi, *Annali di Matematica*, X., pp. 1 and 74 ; Mittag-Leffler, *Annali di Matematica*, XI., p. 65 ; K. Henn, *Math. Annalen*, XXXI. and XXXIII. ; A. Pick, *Wiener Sitz.*, Nov., 1887.

Thursday, April 11th, 1889.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Mr. C. E. Haselfoot was admitted into the Society.

The following communications were made :—

On the Free Vibrations of an Infinite Plate of Homogeneous Isotropic Elastic Matter : Lord Rayleigh, Sec. R.S.

Ueber die constanten Factoren der Thetareihen im allgemeinen Falle $p = 3$: von Felix Klein in Gottingen.

On the generalised Equations of Elasticity, and their application to the Theory of Light : Prof. K. Pearson.

On the Reduction of a complex Quadratic Surd to a Periodic Continued Fraction : Prof. G. B. Mathews.

Construction du Centre de Courbure de la développée de la Courbe de Contour apparent d'une surface que l'on projette orthogonalement sur un plan : Prof. Mannheim.

The President made a few remarks "On an unsymmetric quadrinomial form of the general plane cubic, for which the fundamental invariants are both binomial only."

The Treasurer also made a brief impromptu communication.

The following presents were received :—

"Proceedings of the Royal Society," Vol. XLV., No. 277.

"Proceedings of the Physical Society of London," Vol. X., Part I.

"The Educational Times," for April.

"Proceedings of the Cambridge Philosophical Society," Vol. VI., Part 5.
"Bulletin des Sciences Mathématiques," Tome XIII., February 1889.
"Beiblätter zu den Annalen der Physik und Chemie," Band XIII., Stück 3.
"Jahrbuch über die Fortschritte der Mathematik," Band XVIII., Heft 2.
"American Journal of Mathematics," Vol. XI., No. 3; Baltimore.
"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. IV., Fasc. 11°. Dic. 1888.
"Memorias de la Sociedad Cientifica 'Antonio Alzate,'" Tomo II., No. 6. Diciembre 1888.
"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Num. 77 and 78.
Index to ditto. Two Parts.
"Annali di Matematica," Tome XVI°, Fasc. 4°.
"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 1888, I.—II.; Leipzig, 1889.
"Monographie der Sternhaufen G. C. 4460 und G. C. 1440, sowie einer Sterngruppe bei *o* Piscium," von Bruno Peter; 4to; Leipzig, 1889.
"Über die Affinitätsgrössen organischer Säuren und ihre Beziehungen zur Zusammensetzung und Constitution derselben," von W. Ostwald; 4to; Leipzig, 1889.

On the Free Vibrations of an Infinite Plate of Homogeneous Isotropic Elastic Matter. By Lord RAYLEIGH.

[*Read April 11th*, 1889.]

The solid here contemplated is that bounded by two infinite planes parallel to xy; and the vibrations are supposed to be periodic, not only with respect to the time (e^{ipt}), but also with respect to x and y. The results, so far as thin plates are concerned, have long been known; but the method may not be without interest in view of the difficulties which beset the rigorous treatment of the theory of thin plates, and of the fact that it is not limited to the case of small thickness. A former investigation,* "On Waves propagated along the Plane Surface of an Elastic Solid," may be regarded as a particular case of that now before us.

In conformity with the supposition as to periodicity, we might assume that all the functions concerned involve x and y only through the factors e^{ifx}, e^{igy}. But, by a rotation of the axes, $e^{i(fx+gy)}$ may be replaced by e^{ifx} without loss of generality, and it will considerably

* *Proc. Lond. Math. Soc.*, Vol. XVII., Nov. 1885.

simplify our equations if we limit them to the latter form. Any function of x, y (*e.g.*, the dilatation) may be expanded in a series of such terms as $\cos fx \cos gy$, and this may be resolved into two of the form

$$\cos(fx+gy), \quad \cos(fx-gy).$$

But between these forms there is no essential difference, for on account of the symmetry of the plane we shall have to deal in either case only with $\sqrt{(f^2+g^2)}$. The assumption of proportionality with e^{ifx} is not, however, equivalent to a limitation of the problem to two dimensions, as might at first be supposed; inasmuch as β, the displacement parallel to y, is allowed to remain finite.

If θ be the dilatation, the usual equations are

$$\rho \frac{d^2\alpha}{dt^2} = m \frac{d\theta}{dx} + n\nabla^2\alpha, \text{ \&c.} \quad \text{.......................(1)},$$

in which

$$\theta = \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} \quad \text{..............................(2)},$$

and m, n denote the elastic constants of the material according to Thomson and Tait's notation.*

If α, β, γ all vary as e^{ipt}, equations (1) become

$$m \frac{d\theta}{dx} + n\nabla^2\alpha + \rho p^2\alpha = 0, \text{ \&c.} \quad \text{.....................(3)}.$$

Differentiating equations (3) in order with respect to x, y, z, and adding, we get

$$(\nabla^2 + h^2)\,\theta = 0 \quad \text{..............................(4)},$$

in which

$$h^2 = \rho p^2/(m+n) \text{..............................(5)}.$$

Again, if we put

$$k^2 = \rho p^2/n \quad \text{.................................(6)},$$

equations (3) take the form

$$(\nabla^2 + k^2)\,\alpha = \left(1 - \frac{k^2}{h^2}\right)\frac{d\theta}{dx}, \text{ \&c.} \text{.....................(7)}.$$

A particular solution of (7) is†

$$\alpha = -\frac{1}{h^2}\frac{d\theta}{dx}, \quad \beta = -\frac{1}{h^2}\frac{d\theta}{dy}, \quad \gamma = -\frac{1}{h^2}\frac{d\theta}{dz} \text{........(8)};$$

in order to complete which it is only necessary to add complementary

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* Lamé's constants λ, μ are related to m, n according to $\lambda + \mu = m$, $\mu = n$.

† Lamb "On the Vibrations of an Elastic Sphere," *Math. Soc. Proc.*, May 1882.

terms u, v, w satisfying the equations

$$(\nabla^2+k^2)\,u = 0, \quad (\nabla^2+k^2)\,v = 0, \quad (\nabla^2+k^2)\,w = 0 \qquad (9)$$

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = 0 \qquad (10).$$

According to our present suppositions, x and y are involved only through e^{ifx}, that is, y is not involved at all. Thus

$$\frac{d\theta}{dy} = 0, \qquad \frac{dv}{dy} = 0.$$

The displacement β is thus identical with v, and satisfies the differential equation

$$(\nabla^2+k^2)\,\beta = 0 \qquad (11).$$

Again, in virtue of (9) and (10), we may write

$$u = \frac{d\chi}{dz}, \quad w = -\frac{d\chi}{dx} \qquad (12),$$

where χ is a function of x and z, which satisfies

$$(\nabla^2+k^2)\,\chi = 0 \qquad (13);$$

and
$$\alpha = -\frac{1}{h^2}\,\frac{d\theta}{dx}+\frac{d\chi}{dz}, \quad \gamma = -\frac{1}{h^2}\,\frac{d\theta}{dz}-\frac{d\chi}{dx} \qquad (14).*$$

We have not yet made use of the supposition that x occurs only in the factor e^{ifx}. Under this condition we get from (4)

$$\theta = P\cosh rz + Q\sinh rz \qquad (15),$$

where
$$r^2 = f^2-h^2 \qquad (16);$$

and from (13), (11),

$$\chi = A\sinh sz + B\cosh sz \qquad (17),$$

$$\beta = C\cosh sz + D\sinh sz \qquad (18),$$

where
$$s^2 = f^2-k^2 \qquad (19).$$

The arbitrary quantities P, Q, A, B, C, D may be supposed to include the factors e^{ipt}, e^{ifx}, but are otherwise constants.

The evanescence of the three component stresses at the two bounding surfaces gives, in all, six equations. The components of

* Green, *Camb. Trans.*, 1837; Reprint of Green's Works, p. 261.

tangential stress are, in general, proportional to

$$\frac{d\beta}{dz}+\frac{d\gamma}{dy}, \qquad \frac{d\gamma}{dx}+\frac{d\alpha}{dz}.$$

As regards the first of these, we have at present $d\gamma/dy = 0$; so that the condition to be satisfied at each surface is simply

$$\frac{d\beta}{dz} = 0 \quad \text{.................................(20).}$$

The evanescence of the second tangential stress gives

$$-\frac{2}{h^2}\frac{d^2\theta}{dx\,dz}-\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dz^2} = 0 \quad \text{..................(21).}$$

These equations are to hold good at both surfaces. If we take the origin at the middle of the thickness, the bounding surfaces may be represented by $z = \pm z_1$; and equations (20), (21) must be satisfied by the odd and even functions separately. Thus, from (18), (20),

$$C\sinh sz_1 = 0, \quad D\cosh sz_1 = 0 \quad \text{..................(22),}$$

a pair of equations which may be satisfied in two ways. We may suppose $D = 0$, so that

$$\beta = C\cosh sz \text{...............................(23),}$$

in conjunction with $\sinh sz_1 = 0$ ....(24);

or, on the other hand, $\beta = D\sinh sz$..............................(25),

under the condition $\cosh sz_1 = 0$ ....(26).

During these vibrations the solid is simply sheared. In the vibrations of the first class represented by (23), β is an even function of z, α and γ vanishing. In the vibrations of the second class, β is an odd function of z, and therefore vanishes at the middle surface. The roots of (24) are

$$sz_1 = iq\pi,$$

where q is an integer; so that, by (19),

$$k^2 = f^2+\frac{q^2\pi^2}{z_1^2} \quad \text{..............................(27),}$$

and the stationary vibrations are of the type

$$\beta = \cos pt \cos fx \cos\frac{q\pi z}{z_1} \quad \text{......................(28),}$$

p being given by (6) and (27).

In like manner, for the vibrations of the second class,

$$\beta = \cos pt \cos fx \sin \frac{(q+\frac{1}{2})\,\pi z}{z_1} \quad \text{..................(29)},$$

where

$$k^2 = f^2 + \frac{(q+\frac{1}{2})^2\,\pi^2}{z_1^2} \quad \text{......................(30)}.$$

In (28), (29), we may of course replace $\cos pt$, or $\cos fx$, by $\sin pt$, or $\sin fx$, respectively.*

The kind of vibrations just considered are those for which β is finite, while α and γ vanish. In the second kind of vibrations, β vanishes, so that the motion is strictly in two dimensions. There are four boundary conditions to be satisfied, two derived from (21), and two expressive of the evanescence of the normal stress. The latter condition is that

$$(m-n)\,\theta + 2n\, d\gamma/dz = 0,$$

when $z = \pm z_1$; or, in terms of k^2 and h^2,

$$(k^2-2h^2)\,\theta + 2h^2\, d\gamma/dz = 0 \quad \text{.....................(31)}.$$

Substituting from (14), (15), (17), in (21), (31), we obtain, with use of (16), (19),

$$2ifrh^{-2}\,(P\sinh rz + Q\cosh rz) + (k^2-2f^2)(A\sinh sz + B\cosh sz) = 0 \quad \text{...............(32)},$$

$$(k^2-2f^2)(P\cosh rz + Q\sinh rz) - 2h^2 ifs\,(A\cosh sz + B\sinh sz) = 0 \quad \text{...............(33)}.$$

These equations are to hold when $z = \pm z_1$, and must therefore be true for the odd and even parts separately. Thus

$$2ifrh^{-2}\,P\sinh rz_1 + (k^2-2f^2)\,A\sinh sz_1 = 0 \text{............(34)},$$

$$(k^2-2f^2)\,P\cosh rz_1 - 2h^2 ifs\,A\cosh sz_1 = 0 \text{............(35)};$$

$$2ifrh^{-2}\,Q\cosh rz_1 + (k^2-2f^2)\,B\cosh sz_1 = 0 \text{............(36)},$$

$$(k^2-2f^2)\,Q\sinh rz_1 - 2h^2 ifs\,B\sinh sz_1 = 0 \text{............(37)}.$$

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* In the present investigation the section of the solid perpendicular to y is an infinitely elongated rectangle. It may be worth notice that the corresponding solutions (in which every linear element parallel to the axis moves as a rigid body along its own length) may readily be obtained for cylinders of other sections, *e.g.*, the finite rectangle and the circle. There is complete mathematical analogy with the vibrations of a stretched membrane having the fo[rm] of the section of the cylinder, under the condition that the boundary is free [to m]ove perpendicularly to the plane of the membrane. (*Theory of Sound*, § 227 ., Doc.

It will be seen that in these equations the constants P, A are separated from Q, B. The system can therefore be satisfied in two distinct ways. For the first class of vibrations $Q = 0$, $B = 0$. Equations (36), (37) are thus disposed of; while the first pair serve to determine the ratio $P : A$, and in addition impose a relation between the other quantities. Equations (14) show that θ and α are even functions of z, but that γ is an odd function. In this class of vibrations, therefore, the middle surface remains plane, but undergoes extension.

The frequency equation is found by elimination of $P : A$ between (34), (35):—

$$4f^2rs \sinh rz_1 \cosh sz_1 = (k^2-2f^2)^2 \cosh rz_1 \sinh sz_1\,;$$

or, as it may be written,

$$4f^2rs \tanh rz_1 = (k^2-2f^2)^2 \tanh sz_1 \quad \text{..............(38)}.$$

Again, from (35),

$$\frac{P}{2h^2ifs \cosh sz_1} = \frac{A}{(k^2-2f^2)\cosh rz_1}\,;$$

so that the type of vibration is, by (14),

$$\alpha = e^{ipt}e^{ifx}\left\{2sf^2 \cosh sz_1 \cosh rz + s\,(k^2-2f^2)\cosh rz_1 \cosh sz\right\} \text{...(39)},$$

$$\gamma = -\,e^{ipt}e^{ifx}\left\{2ifrs \cosh sz_1 \sinh rz + if\,(k^2-2f^2)\cosh rz_1 \sinh sz\right\} \text{...(40)}.$$

We may apply these results to the case where the plate is *thin*, so that fz_1 is small. If rz_1, sz_1, in (38), be small, we find

$$(k^2-2f^2)^2 = 4f^2r^2 = 4f^2\,(f^2-h^2),$$

or

$$k^4 = 4f^2\,(k^2-h^2) \quad \text{.........................(41)}.$$

This equation determines k^2, since the ratio h^2/k^2 depends only upon the elastic quality of the material. In terms of m and n, from (5) and (6),

$$k^2 = \frac{4mf^2}{m+n} \text{...............................(42)},$$

or

$$p^2 = \frac{4f^2}{\rho}\,\frac{mn}{m+n} \quad \text{.........................(43)}.$$

At the same time, (39), (40) give approximately

$$\alpha = k^2 s e^{ipt}e^{ifx}, \qquad \gamma = -\,ifsz\,(k^2-2h^2)\,e^{ipt}e^{ifx},$$

or, if we throw out the common factor k^2s,

$\alpha =$ [illegible] e^{ifx}, $\qquad \gamma = -\dfrac{m-n}{m+n}\,ifz\,e^{ipt}e^{ifx}$ ....(44).

This gives the same relation between the principal strains as is obtained in the ordinary theory of thin plates,* viz.,

$$\frac{d\gamma}{dz} = -\frac{m-n}{m+n}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy}\right).$$

A complete discussion of (38) would lead rather far, but we may easily find a second approximation in which the square of z_1 is included. Thus, since

$$\tanh rz_1 = rz_1 \left(1 - \tfrac{1}{3} r^2 z_1^2 + \ldots\right),$$

$$4f^2 r^2 \frac{1-\frac{1}{3}r^2 z_1^2}{1-\frac{1}{3}s^2 z_1^2} = (k^2-2f^2)^2,$$

or

$$4f^2 r^2 \left\{1 - \tfrac{1}{3} z_1^2 (r^2 - s^2)\right\} = (k^2-2f^2)^2;$$

whence, on substitution of the values of r^2 and s^2 from (16), (19),

$$k^4 = 4f^2 (k^2-h^2) \left\{1 - \tfrac{1}{3} z_1^2 (f^2-h^2)\right\} \qquad (45).$$

From the first approximation we know that r^2, or f^2-h^2, is positive. Hence k^2 diminishes with z_1^2, or the pitch falls as the thickness increases. An exception occurs when $r^2 = 0$; but this can happen only when $k^2 = 2f^2 = 2h^2$, or the material is such that $m = n$. If the character of the material be of this description, $k^2 = 2f^2$ satisfies (38), whatever may be the value of z_1. Each lamina parallel to xy vibrates unconstrained by its neighbours, and $\gamma = 0$ throughout.

If the material be incompressible, $h^2 = 0$, and (45) assumes the simplified form

$$k^2 = 4f^2 \left\{1 - \tfrac{1}{3} f^2 z_1^2\right\} \qquad (46).$$

In any of these equations, if we suppose that the functions vary as e^{igy}, as well as e^{ifx}, the generalized result is obtained by merely writing (f^2+g^2) for f^2.

We now pass on to consider the second class of vibrations, for which, in (34), &c., $P = 0$, $A = 0$. Here θ and α are odd functions of z_1, while γ is an even function, so that the middle surface is bent without extension. As regards the equations (36), (37), which involve Q and B, it will be seen that they differ from the first pair of equations involving P and A merely by the interchange everywhere of cosh and sinh. We have, therefore, in place of (38),

$$4f^2 rs \coth rz_1 = (k^2-2f^2)^2 \coth sz_1 \qquad (47);$$

* See, for example, *Proc. Roy. Soc.*, Dec. 1888.

and in place of (39), (40),

$$\alpha = e^{ipt} e^{ifx} \{2sf^2 \sinh sz_1 \sinh rz + s(k^2 - 2f^2) \sinh rz_1 \sinh sz\} \quad \dots(48),$$

$$\gamma = -e^{ipt} e^{ifx} \{2ifrs \sinh sz_1 \cosh rz + if(k^2 - 2f^2) \sinh rz_1 \cosh sz\} \quad \dots(49).$$

If we now introduce the assumption that the plate is thin, we find, by expanding the hyperbolic functions in (47),

$$4f^2(f^2 - k^2)\{1 + \tfrac{1}{3}z_1^2(k^2 - h^2)\} = (k^2 - 2f^2)^2.$$

The first approximation gives $k^2 = 0$, signifying that the notes are infinitely grave. The second approximation is

$$k^4 = \tfrac{4}{3}z_1^2 f^4 (k^2 - h^2) \quad \dots\dots(50),$$

or, in terms of p, m, n, ρ,

$$p^2 = \frac{mn}{m+n} \frac{4f^4 z_1^2}{3\rho} \quad \dots\dots(51).$$

Again, if we drop out a common factor $(k^2 r z_1)$, (48), (49) take the forms

$$\alpha = f^2 z \, e^{ipt} e^{ifx}, \quad \gamma = if e^{ipt} e^{ifx} \quad \dots\dots(52).$$

Hence $\alpha = -z \, d\gamma/dx$, signifying that to this order of approximation every line originally perpendicular to the middle surface retains its straightness and perpendicularity during the vibrations.

The third approximation to the value of k^2 from (47) gives

$$p^2 = \frac{mn}{m+n} \frac{4f^4 z^2}{3\rho} \left\{1 - f^2 z_1^2 \left[\frac{4m}{3(m+n)} + \frac{7}{15}\right]\right\} \quad \dots\dots(53);$$

so that, when the thickness is increased beyond a certain point, the rise of pitch begins to be less rapid than according to the second approximation (51).

When z_1 is infinitely great, we get, from (38) or (47),

$$4f^2 rs = (k^2 - 2f^2)^2 \quad \dots\dots(54),^*$$

the equation considered in the paper already referred to upon surface-waves.

From (43), (53) we learn that p^2 is positive, or the equilibrium is stable, so long as m is positive. On the other hand, it was proved by Green many years ago that a solid body would be unstable if m were less than $\frac{1}{3}n$, $m - \frac{1}{3}n$ being in fact the dilatation modulus. The reconciliation of these apparently contradictory results depends upon

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* This is upon the supposition that r and s are real. In the contrary case the equation would have no definite limit.

principles similar to those recently applied by Sir W. Thomson,* to show that a solid, every part of the boundary of which is held fixed, is stable, so long as m is greater than $-n$, and this in spite of the fact that, if the boundary were freed, the solid would at once collapse or expand indefinitely. In the present case of an infinite slab, the assumption that the displacements are periodic with respect to x and y is tantamount to the imposition of a constraint at infinity, rendering stability possible under circumstances which would otherwise lead to indefinite collapse or expansion of the medium.

The general expression for the energy of a strained isotropic solid is†

$$2w = (m+n)\,(\mathrm{e}^2+\mathrm{f}^2+\mathrm{g}^2)+2\,(m-n)(\mathrm{fg}+\mathrm{ge}+\mathrm{ef})+n\,(a^2+b^2+c^2) \quad\text{............(55)},$$

e, f, g being the principal extensions; a, b, c the shears relatively to the coordinate axes. Since e, f, g may vanish, it is clear that the stability of the medium requires that n be positive; and again, since a, b, c may all vanish, the terms in e, f, g must of themselves be positive in all cases that may arise.

Thus, leaving out a, b, c, we write

$$2w = (3m-n)\,(\mathrm{e}^2+\mathrm{f}^2+\mathrm{g}^2)+(n-m)\,\{(\mathrm{e}-\mathrm{f})^2+(\mathrm{f}-\mathrm{g})^2+(\mathrm{g}-\mathrm{e})^2\} \quad\text{............(56)},$$

from which it follows that, if $n > m > \frac{1}{3}n$, the equilibrium is stable. If, however, $m < \frac{1}{3}n$, it will be possible to make w negative by taking $\mathrm{e} = \mathrm{f} = \mathrm{g}$. If $m > n$, the equilibrium is stable, as may be seen by writing $2w$ in the form

$$2w = (m-n)(\mathrm{e}+\mathrm{f}+\mathrm{g})^2+2n\,(\mathrm{e}^2+\mathrm{f}^2+\mathrm{g}^2) \quad\text{..............(57)}.$$

Hence, if there be no limitation on the strains, the necessary and sufficient conditions of stability are that n should be positive and m greater than $\frac{1}{3}n$.

But now suppose that the strains are limited to be in two dimensions, so that (for example) $\mathrm{g} = 0$. The supposition $\mathrm{e} = \mathrm{f} = \mathrm{g}$ is then not admissible, and the criterion of stability is altered. We have

$$\begin{aligned} 2w &= (m+n)(\mathrm{e}^2+\mathrm{f}^2)+2\,(m-n)\,\mathrm{ef} \\ &= (n-m)(\mathrm{e}-\mathrm{f})^2+2m\,(\mathrm{e}^2+\mathrm{f}^2) \quad\text{.....................(58)}. \end{aligned}$$

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* *Phil. Mag.*, Nov., 1888.

† Thomson and Tait's *Natural Philosophy*, § 695.

This shows that there is stability if m be positive and less than n, and instability if m be negative. That the equilibrium is stable if m be greater than n is shown, as in (57), by putting $2w$ into the form

$$2w = (m-n)(\mathrm{e}+\mathrm{f})^2 + 2n(\mathrm{e}^2+\mathrm{f}^2) \quad \text{.................} (59).$$

Hence, under the limitation $\mathrm{g} = 0$, the necessary and sufficient conditions of stability are that n and m be positive.

Comparing the results, we see that, as m diminishes, instability sets in when $m = \frac{1}{3}n$, if the boundary be free; when $m = 0$, if (as virtually in our present problem) the strains be limited to two dimensions; when $m = -n$, if the boundary be everywhere held fast.

I have endeavoured to investigate the two-dimensional free vibrations of an infinitely long cylindrical shell directly from the fundamental equations, as in the foregoing theory of the plane plate. The preliminary analysis is simple, and there is no difficulty in obtaining the solutions analogous to (42). If a be the radius of the cylinder, and the wave-length measured round the circumference be $2\pi/f$, we have

$$k^2a^2 = \frac{4m}{m+n}(f^2a^2+1) \quad \text{.....................} (60),$$

and

$$p^2 = \frac{4(f^2a^2+1)}{\rho a^2}\,\frac{mn}{m+n} \quad \text{.....................} (61).$$

But this solution is much more readily obtained by the special methods applicable to thin plates, as to the legitimacy of which for this purpose there can be no question. And if, in order to investigate the flexional vibrations of the shell, we retain the lower powers of the thickness, the reduction of the resulting determinant becomes a very complicated affair. I have not succeeded in verifying by a rigorous application of this method the equation analogous to (51), viz.:

$$p^2 = \frac{mn}{m+n}\,\frac{4f^2z_1^2}{3\rho a^4}\,\frac{(f^2a^2-1)^2}{f^2a^2+1} \quad \text{...................} (62),$$

$2z_1$ being the thickness, and as before fa the number of wave-lengths in the circumference. Putting $a = \infty$, we fall back, of course, upon the formulæ for the plane plate.

Ueber die constanten Factoren der Thetareihen im allgemeinen Falle $p = 3$. Von Felix Klein, *in* Göttingen.

[*Read April* 11*th*, 1889.]

Sei $F(x_1 x_2 x_3) = 0$ eine allgemeine Curve 4[ter] Ordnung der Ebene, w_1, w_2, w_3 seien die zugehörigen überall endlichen Integrale

$$w_1 = \int x_1\, d\omega, \quad w_2 = \int x_2\, d\omega, \quad w_3 = \int x_3\, d\omega,$$

wo in bekannter Weise

$$d\omega = \frac{|\, c x dx \,|}{\Sigma c_i F_i}.$$

Unter Zugrundelegung einer bestimmten Zerschneidung der zur Curve gehörigen Riemann'schen Fläche bilden wir die 64 zu unterscheidenden Thetareihen, substituiren in dieselben die w, und fragen, *welches die Anfangsglieder der Potenzentwickelungen sind, welche die Theta hinsichtlich der w_1, w_2, w_3 gestatten.* Diese Frage ist seither, so viel ich weiss, noch nicht beantwortet worden; ich erlaube mir also mein bezügliches Resultat mitzutheilen.

Sei zuvörderst ϑ eine *gerade* Thetafunction. Dann verlangt unsere Frage, den Werth von $\vartheta(0, 0, 0)$ anzugeben. Die allgemeine Form der in dieser Hinsicht anzustrebenden Formel kann als bekannt gelten; wir werden haben:

$$(1) \quad \vartheta(0, 0, 0) = c\,\sqrt{p_{123}} \cdot \sqrt[8]{G},$$

wo c eine numerische Constante ist (mit deren Bestimmung wir uns hier nicht aufhalten wollen), p_{123} die Determinante der drei ersten Perioden ist, welche w_1, w_2, w_3 hinsichtlich des gewählten Querschnittsystems besitzen:

$$p_{123} = \begin{vmatrix} \omega_{11}, & \omega_{12}, & \omega_{13} \\ \omega_{21}, & \omega_{22}, & \omega_{23} \\ \omega_{31}, & \omega_{32}, & \omega_{33} \end{vmatrix},$$

G aber eine irrationale Invariante von $F(x_1 x_2 x_3)$ vorstellt, die in den Coefficienten von F den Grad 12 hat. Es handelt sich uns um die Bestimmung dieses G. Hierüber ist nun Folgendes zu bemerken:

Jedem einzelnen der geraden ϑ entsprechend lässt sich die Curve vierter Ordnung bekanntermassen in der zuerst von *Hesse* untersuchten

Weise auf die Kegelspitzencurve eines Netzes von Flächen zweiter Ordnung beziehen:

$$x_1 . \Sigma \alpha_{ik} z_i z_k + x_2 . \Sigma \beta_{ik} z_i z_k + x_3 . \Sigma \gamma_{ik} z_i z_k = 0,$$

wobei die Gleichung der Curve vierter Ordnung die Gestalt einer symmetrischen viergliedrigen Determinante annimmt:

$$\Pi = \begin{vmatrix} \pi_{11} \dots \pi_{14} \\ \dots \dots \dots \\ \pi_{41} \dots \pi_{44} \end{vmatrix} = 0,$$

wo
$$\pi_{ik} = x_1 . \alpha_{ik} + x_2 . \beta_{ik} + x_3 . \gamma_{ik}.$$

Die Invariante G ist nun nichts Anderes als die Discriminante dieses Netzes

$$D(\alpha_{ik}, \beta_{ik}, \gamma_{ik}),$$

d. h., sie ist eine homogene Function sechszehnten Grades der α_{ik}, wie der β_{ik} und der γ_{ik}, welche, gleich Null gesetzt, die Bedingung ergebt, unter der von den acht Grundpuncten des Netzes zwei zusammen fallen. In der That ist diese Discriminante eine Combinante der drei quaternären Formen $\Sigma \alpha_{ik} z_i z_k$, $\Sigma \beta_{ik} z_i z_k$, $\Sigma \gamma_{ik} z_i z_k$, und als solche eine ganze (algebraische aber nicht rationale) Function zwölften Grades der Coëffic enten von Π.

Sei jetzt ferner ϑ eine ungerade Thetafunction, $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$ die ihr zugehörige Doppeltangente der Curve vierter Ordnung $F = 0$. So kann wieder folgender Ansatz als bekannt gelten:

$$(2) \quad \left(\frac{\delta \vartheta}{\delta w_1}\right)_{000} . w_1 + \left(\frac{\delta \vartheta}{\delta w_2}\right)_{000} . w_2 + \left(\frac{\delta \vartheta}{\delta w_3}\right)_{000} . w_3$$
$$= \gamma . \sqrt{p_{123}} . \sqrt[8]{\Gamma} . (u_1 w_1 + u_2 w_2 + u_3 w_3);$$

hier ist γ eine numerische Constante, die wir unbestimmt lassen, Γ die von uns festzulegende Invariante. Wir setzen $F = 0$ in die bekannte Gleichungsform

$$u . \Phi - \Omega^2 = 0,$$

wo $\Omega = 0$ irgend ein Kegelschnitt ist, der durch die Berührungspuncte der Doppeltangente durchläuft, Φ eine zugehörige Form dritten Grades der x_1, x_2, x_3. Man kann diese Gleichung mit Herrn *Geiser* dahin interpretiren, dass die Curve vierter Ordnung vermöge derselben als der scheinbare Umriss definirt wird, den die Fläche dritter Ordnung:

$$(u_1 x_1 + u_2 x_2 + u_3 x_3) . x_4^2 - 2\Omega(x_1 \, x_2 \, x_3) . x_4 + \Phi \, (x_1 \, x_2 \, x_3) = 0,$$

vom Puncte $x_1 = x_2 = x_3 = 0$ aus bei Projection auf die Ebene $x_4 = 0$ darbietet. *Ich sage nun, dass unser* Γ *einfach die Discriminante dieser Fläche dritter Ordnung ist, d. h., diejenige ganze rationale homogene Function* 32ten *Grades der Coefficienten der Fläche, welche gleich Null gesetzt besagt, dass die Fläche einen Doppelpunct besitzt.* In der That wird diese Discriminante so geschrieben werden können, dass sie als simultane Invariante der beiden ternären Formen

$$F = u_x . \Phi - \Omega^2$$

und ϕ erscheint, vom zwölften Grade in den Coëfficienten von F, vom achten Grade in den Coefficienten von Φ, genau wie es die Rücksicht auf die Homogeneïtat in Formel (2) verlangt.

On the Reduction of a Complex Quadratic Surd to a Periodic Continued Fraction. By G. B. MATHEWS.

[*Read April* 11*th*, 1889.]

The square root of a complex integer may be reduced to a periodic chain-fraction by a process very similar to that employed for an ordinary quadratic surd. Dirichlet alludes to this, in passing, in his memoir on quadratic forms with complex coefficients (*Crelle*, t. xxiv.); but as he does not enter into any details with regard to the algorithm, and since no attempt appears to have been made to construct a complex "Canon Pellianus," the following table of results may be useful. In the column on the left will be found all the complex not-square integers $a+bi$, of which the norm does not exceed 100, and in which a, b are both positive; on the right-hand, opposite to $a+bi$, are entered the partial quotients belonging to the chain-fraction expansion of $\sqrt{a+bi}$. A semicolon has been placed before the first term of the period, so that, for instance, $2+2i$; 4, $4+4i$ means

$$2+2i+\frac{1}{4+}\;\frac{1}{(4+4i)}+\frac{1}{4+}\;\frac{1}{(4+4i)}+\&c.$$

Each stage of the process of reduction is of the type

$$z_n = \frac{E_n+\sqrt{A}}{D_n} = a_n + \frac{1}{z_{n+1}},$$

where a_n is a complex integer, determined so that

$$Nm\,(z_n - a_n) < \tfrac{1}{2}.$$

As in the ordinary theory (*cf.* Serret, *Algèbre Supérieure*, t. i., c. 2) all the quantities E_n, D_n are integers, and we have

$$E_n^2 + D_{n-1} D_n = A,$$

$$E_n + E_{n-1} = a_{n-1} D_{n-1}.$$

Inspection of the table suggests many curious theorems analogous to those of the ordinary theory: for instance, in all the cases examined the period begins with the second partial quotient; and, if a be the first quotient, then the last is $2a - e - e'i$, where e, e' are 0 or 1. It would be interesting to account for this independently of the theory of reduced quadratic forms; the analogy of the ordinary theory soon breaks down because the restriction that $a, a_1, a_2 \ldots$ are *positive* integers does not hold good.

It is not difficult, however, to show that the expansion is really periodic: namely, it follows from the way in which a_{n-1} is determined that

$$Nm\left(\frac{E_{n-1} - a_{n-1} D_{n-1} + \sqrt{A}}{D_{n-1}}\right) < \tfrac{1}{2},$$

i.e.,
$$Nm\,(\sqrt{A} - E_n) < \tfrac{1}{2} Nm\,(D_{n-1}),$$

therefore
$$\text{mod.}\,(\sqrt{A} - E_n) < \frac{1}{\sqrt{2}}\,\text{mod.}\,(D_{n-1}),$$

therefore
$$\text{mod.}\,(\sqrt{A} + E_n) > \sqrt{2}\,\text{mod.}\,(D_n).$$

Now, suppose
$$\text{mod.}\,D_{n-1} < \mu\,\text{mod.}\,\sqrt{A};$$

then
$$\text{mod.}\,E_n < \text{mod.}\,\sqrt{A} + \text{mod.}\,(\sqrt{A} - E_n)$$
$$< \left(1 + \frac{\mu}{\sqrt{2}}\right)\text{mod.}\,\sqrt{A},$$

and
$$\sqrt{2}\,\text{mod.}\,D_n < \text{mod.}\,\sqrt{A} + \text{mod.}\,E_n$$
$$< \left(2 + \frac{\mu}{\sqrt{2}}\right)\text{mod.}\,\sqrt{A},$$

therefore
$$\text{mod.}\,D_n < \left(\frac{\mu}{2} + \sqrt{2}\right)\text{mod.}\,\sqrt{A}.$$

This agrees with the inequality supposed satisfied by mod. D_{n-1}, if

$$\mu = \frac{\mu}{2} + \sqrt{2},$$

or $$\mu = 2\sqrt{2}.$$

Hence, if $$\text{mod. } D_{n-1} < 2\sqrt{2} \text{ mod. } \sqrt{A},$$

it follows that $$\text{mod. } D_n < 2\sqrt{2} \text{ mod. } \sqrt{A},$$

$$\text{mod. } E_n < 3 \text{ mod. } \sqrt{A},$$

and $$\text{mod. } a_n < \text{mod. } E_n + \text{mod. } E_{n-1} < 6 \text{ mod. } \sqrt{A}.$$

We conclude, therefore, by induction, that the moduli of the quantities E_n, D_n, a_n cannot exceed certain finite limits, and hence the expansion must be periodic.

$1+i$	$1;\ -2i,\ 2$
$1+2i$	$1+i;\ 2+2i$
$1+3i$	$1+i;\ 2,\ 2+2i$
$1+4i$	$2+i;\ -2-i,\ 4+2i$
$1+5i$	$2+i;\ -1-2i,\ -4i,\ -1-2i,\ 4+2i$
$1+6i$	$2+2i;\ -1+2i,\ 2-i,\ -1+2i,\ 4+4i$
$1+7i$	$2+2i;\ 4i,\ 4+4i$
$1+8i$	$2+2i;\ 4+4i$
$1+9i$	$2+2i;\ 4,\ 4+4i$
$1+10i$	$2+2i;\ 3-i,\ -2,\ 3i,\ -2+3i,\ -1+3i,\ 2,\ 2-i,\ 4+4i$
$2+i$	$1;\ 1-i,\ 2$
$2+2i$	$2+i;\ -1+i,\ -3-i,\ 1-i,\ 3+i$
$2+3i$	$2+i;\ -3+i,\ 4+2i$
$2+4i$	$2+i;\ -4-2i,\ 4+2i$
$2+5i$	$2+i;\ -1-3i,\ 4+2i$
$2+6i$	$2+i;\ -2i,\ 4+2i$
$2+7i$	$2+2i;\ 1+2i,\ -1-3i,\ 1+2i,\ 4+4i$
$2+8i$	$2+2i;\ 2+2i,\ 4+4i$
$2+9i$	$2+2i;\ 2+i,\ 1+i,\ 2+i,\ 4+4i$
$3+i$	$2;\ -2-2i,\ 4$
$3+2i$	$2+i;\ -1+2i,\ 4+2i$
$3+3i$	$2+i;\ -2+4i,\ 4+2i$
$3+5i$	$2+i;\ 2-4i,\ 4+2i$
$3+6i$	$2+i;\ 1-2i,\ 4+2i$
$3+7i$	$2+2i;\ 1+i,\ -2i,\ -4,\ 1-2i,\ -1+i,\ -4-3i,\ -1-i,\ 2i,\ 4,$ $-1+2i,\ 1-i,\ 4+3i$

$3+8i$	$2+2i;\ 1+i,\ 2-i,\ -3-2i,\ 3-i,\ -1+i,\ -4-3i,\ -1-i,\ -2+i,\ 3+2i,\ -3+i,\ 1-i,\ 4+3i$
$3+9i$	$2+2i;\ 2+i,\ -2+2i,\ 2+i,\ 4+4i$
$4+i$	$2;\ -4i,\ 4$
$4+2i$	$2+i;\ 2i,\ 4+2i$
$4+3i$	$2+i;\ 1+3i,\ 4+2i$
$4+4i$	$2+i;\ 4+2i$
$4+5i$	$2+i;\ 3-i,\ 4+2i$
$4+6i$	$2+i;\ 2-i,\ -2+2i,\ 2-i,\ 4+2i$
$4+7i$	$2+i;\ 1-i,\ 4+2i$
$4+8i$	$3+2i;\ -1+i,\ -5-3i,\ 1-i,\ 5+3i$
$4+9i$	$3+2i;\ -2+i,\ 2-2i,\ -1-i,\ -3+5i,\ 1+2i,\ 2+2i,\ -1+i,\ -5-3i,\ 2-i,\ -2+2i,\ 1+i,\ 3-5i,\ -1-2i,\ -2-2i,\ 1-i,\ 5+3i$
$5+i$	$2;\ 2-2i,\ 4$
$5+2i$	$2;\ 1-2i,\ -3i,\ 1-2i,\ 4$
$5+3i$	$3+i;\ -1+i,\ -1+2i,\ 2-5i,\ -1-i,\ 2+i,\ 5+2i$
$5+4i$	$2+i;\ 2+i,\ 4+2i$
$5+5i$	$2+i;\ 2,\ 4+2i$
$5+6i$	$3+i;\ -2-i,\ 2-3i,\ -2-i,\ -6-2i,\ 2+i,\ -2+3i,\ 2+i,\ 6+2i$
$5+7i$	$2+i;\ 1-i,\ 2-2i,\ -3,\ 2+4i,\ -3,\ 2-2i,\ 1-i,\ 4+2i$
$5+8i$	$3+i;\ -1-i,\ 2i,\ 3-i,\ -1,\ -1-i,\ -2+i,\ -3i,\ 1-i,\ 5+2i$
$6+i$	$2;\ 2-i,\ -3-3i,\ 2-i,\ 4$
$6+2i$	$2;\ 1-i,\ 4$
$6+3i$	$3+i;\ -1+i,\ -5-i,\ 1-i,\ 5+i$
$6+4i$	$3+i;\ -2+i,\ 6+2i$
$6+5i$	$3+i;\ -3,\ 1-i,\ 1+i,\ 3-i,\ 1+2i,\ 2+i,\ -2,\ -5-2i,\ 3,\ -1+i,\ -1-i,\ -3+i,\ -1-2i,\ -2-i,\ 2,\ 5+2i$
$6+6i$	$3+i;\ -3-i,\ 6+2i$
$6+7i$	$3+i;\ -2-2i,\ 6+2i$
$7+i$	$3;\ -2-i,\ -2+2i,\ -2-i,\ 6$
$7+2i$	$3;\ -1-2i,\ -1-i,\ 2i,\ 1-2i,\ 1-i,\ 5$
$7+3i$	$3+i;\ -1+2i,\ 2i,\ -2-3i,\ -1-3i,\ 1-i,\ 5+i$
$7+4i$	$3+i;\ -2+2i,\ 6+2i$
$7+5i$	$3+i;\ -4+2i,\ 6+2i$
$7+6i$	$3+i;\ -6-2i,\ 6+2i$
$7+7i$	$3+i;\ -2-4i,\ 6+2i$

$8+i$	$3;\ -3-3i,\ 6$
$8+2i$	$3;\ -1-2i,\ 2i,\ -1-2i,\ 6$
$8+3i$	$3+i;\ -1+2i,\ 2,\ 1-6i,\ -2-i,\ -2i,\ 6+i$
$8+4i$	$3+i;\ -1+3i,\ 6+2i$
$8+5i$	$3+i;\ -2+6i,\ 6+2i$
$9+i$	$3;\ -6i,\ 6$
$9+2i$	$3;\ -3i,\ 6$
$9+3i$	$3;\ -2i,\ 6$
$9+4i$	$3+i;\ 1+3i,\ -2+i,\ 4+i,\ -2+i,\ 1+3i,\ 6+2i$
$10+i$	$3;\ 3-3i,\ 6$

Construction du centre de courbure de la développée de la courbe de contour apparent d'une surface que l'on projette orthogonalement sur un plan. By Prof. A. MANNHEIM.

[*Read April 11th*, 1889.]

Dans une communication sur les surfaces parallèles, que j'ai eu l'honneur de faire à la Société Mathématique de Londres,* je suis arrivé à une expression du rayon de courbure de la développée de la courbe de contour apparent d'une surface. La recherche de cette expression est un de ces problèmes dont la solution dépend des infiniment petits du 3[e] ordre et que, le premier, j'ai traités géométriquement.

Dans la même communication, j'ai fait connaître aussi une autre expression de ce même rayon de courbure, et j'ai annoncé que je montrerais comment on peut l'obtenir directement.

C'est cette démonstration directe que je vais exposer aujourd'hui, en faisant usage de deux droites dont j'ai déjà eu plusieurs fois l'occasion de prouver l'utilité.†

Il s'agit donc moins ici du problème particulier que je traite dans

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* Voir les *Proceedings*, Vol. XII., No. 177.
† Voir *Comptes rendus de l'Académie des Sciences*, Séances des 22 Mars 1875 et 6 Mars 1876.

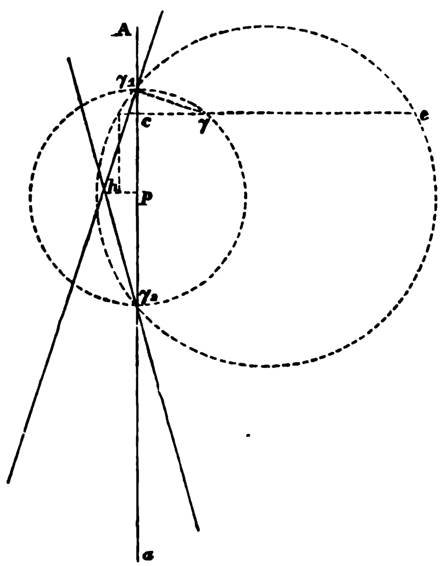
cette note que de la méthode géométrique dont je veux faire ressortir une fois de plus l'élégante simplicité.

En effet, à l'aide de ces deux droites on peut résoudre avec la plus grande facilité des questions pour lesquelles la méthode analytique, au milieu de ses longues formules, ne laisse pas apercevoir les éléments simples qui doivent seuls subsister dans toute construction.

Les deux droites dont je viens de parler sont de ces éléments simples. Rappelons d'abord leur origine.

A partir d'un point a sur une surface donnée (S) traçons des courbes tangentes entr'elles. On sait que les normalies à (S) dont ces courbes sont les directrices sont osculatrices entr'elles aux deux centres de courbure principaux γ_1, γ_2 situés sur la droite A normale en a à (S). Ces normalies ont donc en γ_1 et γ_2 les mêmes indicatrices.

Les asymptotes de ces indicatrices sont alors les mêmes pour toutes ces normalies et, comme A est l'une de ces asymptotes, les autres asymptotes communes sont deux droites issues respectivement des points γ_1 et γ_2. Ce sont là les deux droites que je vais employer. Je puis me les donner d'avance car, si, pour définir les éléments du 3^e ordre de la surface (S), on suppose connues les droites de courbure des nappes de la développée de (S), j'ai fait voir il y a longtemps déjà comment on peut construire ces deux droites.*



* *Comptes rendus de l'Académie des Sciences*, Séance du 1er Mars 1875.

Supposons que le plan de projection contienne la normale A (Fig. 1); la courbe de contour apparent de (S) sur ce plan est la trace d'un cylindre circonscrit à cette surface et dont les génératrices sont perpendiculaires au plan de projection. Le centre de courbure de la courbe de contour apparent qui correspond au point a, s'obtient de la manière suivante :* *Les points γ_1, γ_2 sur A étant les centres de courbure principaux de (S), on décrit sur $\gamma_1\gamma_2$ comme diamètre une circonférence de cercle. Du point γ_1, extrémité du rayon de courbure principal maximum, on mène la droite $\gamma_1\gamma$ qui fait avec A un angle égal à l'angle des projetantes avec le grand axe de l'indicatrice de (S) en* a, *cette droite rencontre la circonférence au point γ. La projection de ce point sur A est le centre de courbure* c *cherché.*

La développée de la courbe de contour apparent de (S) touche A au point c; ce que nous nous proposons de déterminer, c'est le centre de courbure de cette développée qui correspond à ce point c.

Le cylindre circonscrit à (S), et dont les génératrices sont perpendiculaires au plan de projection, touche cette surface suivant une courbe que je prends pour directrice d'une normalie à (S). Les génératrices de cette normalie sont parallèles au plan de projection, et leurs projections sur ce plan sont des tangentes à la développée de la courbe de contour apparent de (S).

Puisque les génératrices de cette normalie sont parallèles au plan de projection, il existe le long de A un paraboloïde osculateur de cette normalie. Ce paraboloïde osculateur a pour plan directeur le plan de projection, et pour directrices les asymptotes des indicatrices de la normalie en γ_1 et γ_2, qui ne sont pas la droite A. Ces deux asymptotes sont les deux droites dont j'ai parlé plus haut et qui conduisent immédiatement à la solution de notre problème.

Supposons que leurs projections soient $\gamma_1 h$ et $\gamma_2 h$. Le paraboloïde osculateur de la normalie a pour contour apparent sur le plan de projection une parabole qui est, au point c, osculatrice de la développée de la courbe de contour apparent de (S).

On est donc ramené à chercher le rayon de courbure de cette parabole pour le point c. Cette parabole est tangente en c à A, et elle a pour tangentes $\gamma_1 h$ et $\gamma_2 h$. Dans ces conditions, pour déterminer au point c son rayon de courbure ρ, on a cette formule :†

$$\frac{1}{c\gamma_1}+\frac{1}{c\gamma_2}=\frac{2}{\rho}\left(\frac{1}{\operatorname{tang} c\gamma_1 h}+\frac{1}{\operatorname{tang} c\gamma_2 h}\right),$$

* Voir mon *Cours de Géométrie descriptive*, 2e édition, p. 321.

† *Comptes rendus de l'Académie des Sciences*, Séance du 15e Mars 1875.

ou, en abaissant la perpendiculaire hp sur A :

$$\frac{\gamma_1\gamma_2}{c\gamma_1 \times c\gamma_2} = \frac{2}{\rho}\left(\frac{\gamma_1 p}{ph} + \frac{\gamma_2 p}{ph}\right),$$

$$\frac{1}{c\gamma_1 \times c\gamma_2} = \frac{2}{\rho \times ph}.$$

D'après cela, pour déterminer le centre de courbure demandé, on a la construction suivante :

Sur la perpendiculaire ce à A, qui est la normale à la développée de la courbe de contour apparent de (S), on projette le point milieu de hp. *La circonférence qui passe par le point ainsi obtenu et par les points γ_1, γ_2 coupe la normale ce au centre de courbure demandé* e.

Le point c est sur A le point central de la normalie. Aux points γ_1, γ_2 les plans tangents à cette normalie sont rectangulaires. Le produit $c\gamma_1 \times c\gamma_2$ est alors égal au carré du paramètre de distribution des plans tangents à cette normalie pour la droite A. En appelant K ce paramètre, la formule précédente devient :

$$\rho = \frac{2K^2}{ph}.$$

On retrouve ainsi l'expression que je n'avais fait qu'indiquer jadis et dont j'ai parlé en commençant cette courte note.

Remarque.—Dans mon travail sur les surfaces parallèles, j'ai démontré que : *les centres de courbure géodésique des courbes à courbure normale constante, tangentes aux traces d'une normalie sur des surfaces parallèles entr'elles, sont sur une même droite I.*

Cette droite I est à la fois une génératrice du paraboloïde des normales à la normalie (dont j'ai parlé précédemment), et une génératrice du paraboloïde dont un plan directeur est perpendiculaire à A, et dont les directrices sont les deux droites projetées en $\gamma_1 h$ et $\gamma_2 h$.

La droite I appartenant au paraboloïde des normales à la normalie est parallèle au plan central de cette surface. Comme ce plan central passe par A et est perpendiculaire au plan de projection, la droite I se projette suivant une parallèle à A.

La droite I, étant une génératrice de l'autre paraboloïde, sa projection passe par le point h. Donc : *La droite I se projette suivant la parallèle menée de* h *à la droite A.*

Thursday, May 9th, 1889.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

The following communications were made:—

On the Solution in Integers of Equations of the form $x^3+y^3+Az^3=0$: S. Roberts, F.R.S.

On the Concomitants of k-ary Quantics: W. J. C. Sharp, M.A.

On the Motion of an Elastic Solid strained by Extraneous Forces: Signor Betti.

Note on the G-function in an Elliptic Transformation Annihilator: J. Griffiths, M.A.

On Cyclotomic Functions, § iii. The Cyclotomics belonging to the f-nomial periods of the p^{th} Roots of Unity: Prof. Lloyd Tanner.

On the Complete Elliptic Integrals, K, E, G, I: Dr. J. Kleiber.

The following presents were received:—

A Cabinet Likeness of Dr. G. J. Allman, F.R.S.

"Educational Times," for May.

"Annals of Mathematics," Vol. IV., No. 5, Oct. 1888; Virginia.

"Bulletin des Sciences Mathématiques," Tome XIII., March and April.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," XXXVIII.—LII.

"Atti della Reale Accademia dei Lincei — Rendiconti," Vol. IV., Fasc. 12; Vol. V., Fasc. 1—3; Roma, 1888, 1889.

"Journal für die reine und angewandte Mathematik," Band 104, Heft III.

"Die Rotationsmomente der Beugemuskeln am Ellbogengelenk des Menschen," von W. Braune und O. Fisher; large 8vo; Leipzig, 1889.

"Die Neuroblasten und deren Entstehung im Embryonalen Mark," von Wilhelm His; large 8vo; Leipzig, 1889.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Nos. 79 and 80.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. IX., No. 1; Coimbra, 1889.

"Reale Istituto Lombardo di Scienze e Lettere—Rendiconti," Serie II., Vol. XX., 8vo; Milano, &c., 1887.

"Memorie del Reale Istituto Lombardo," Vol. XVI., Fasc. II., 4to; Milano, &c., 1888.

On the Motion of an Elastic Solid strained by Extraneous Forces. By Enrico Betti.

[*Read May 9th*, 1889.]

Let L, M, N be the forces at the boundary of an elastic solid, and u, v, w be the components of displacements, which bring it to equilibrium. The equations of equilibrium are, in the internal space S—

$$\left.\begin{aligned} 0 &= \frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z} \\ 0 &= \frac{\partial U}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial S}{\partial z} \\ 0 &= \frac{\partial T}{\partial x}+\frac{\partial S}{\partial y}+\frac{\partial R}{\partial z} \end{aligned}\right\} \quad \dots\dots\dots\dots (1),$$

and at the boundary σ—

$$\left.\begin{aligned} L &= P\alpha+U\beta+T\gamma \\ M &= U\alpha+Q\beta+S\gamma \\ N &= T\alpha+S\beta+R\gamma \end{aligned}\right\} \quad \dots\dots\dots\dots (2),$$

where P, Q, R denote normal components of pull on interfaces respectively perpendicular to x, y, z; and S, T, U the tangential components.

Multiplying the first of the equations (1) by v, the second by u, and subtracting one from the other, and integrating throughout all the space S, we have

$$0 = \int_\sigma (Lv-Mu)\,d\sigma$$

$$+\int_S \left[P\frac{\partial v}{\partial x}-Q\frac{\partial u}{\partial y}+U\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)+T\frac{\partial v}{\partial z}-S\frac{\partial u}{\partial z}\right] dS \quad \dots\dots (3).$$

Let us take another system of displacements, u', v', w', for which, at the boundary σ,

$$u' = u, \quad v' = v, \quad w' = w \quad \dots\dots\dots\dots (4),$$

and in all the space S,

$$\left.\begin{aligned}
&\frac{\partial u'}{\partial x}=\frac{\partial u}{\partial x},\quad \frac{\partial v'}{\partial y}=\frac{\partial v}{\partial y},\quad \frac{\partial w'}{\partial z}=\frac{\partial w}{\partial z}\\
&\frac{\partial u'}{\partial y}=\frac{\partial u}{\partial y}+\omega_3,\quad \frac{\partial v'}{\partial z}=\frac{\partial v}{\partial z}+\omega_1,\quad \frac{\partial w'}{\partial x}=\frac{\partial w}{\partial x}+\omega_2\\
&\frac{\partial v'}{\partial x}=\frac{\partial v}{\partial x}-\omega_3,\quad \frac{\partial w'}{\partial y}=\frac{\partial w}{\partial y}-\omega_1,\quad \frac{\partial u'}{\partial z}=\frac{\partial u}{\partial z}-\omega_2
\end{aligned}\right\} \quad \ldots\ldots\ldots(5).$$

In order that the solid may remain in equilibrium with these displacements, it is necessary that ω_1, ω_2, ω_3 should be constants in all the space S. Therefore, if they are different from zero only in one part S' of S, we shall have motion, and instead of the equations (1) we shall have the following:—

$$\left.\begin{aligned}
\rho\frac{d^2u}{dt^2}&=\frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}\\
\rho\frac{d^2v}{dt^2}&=\frac{\partial U}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial S}{\partial z}\\
\rho\frac{d^2w}{dt^2}&=\frac{\partial T}{\partial x}+\frac{\partial S}{\partial y}+\frac{\partial R}{\partial z}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where ρ denotes the density. From the equations (6) we deduce, observing the equations (4) and (5),

$$\int_S \rho\left(v'\frac{d^2u'}{dt^2}-u'\frac{d^2v'}{dt^2}\right)dS$$

$$=\int_\sigma (Lv-Mu)\,d\sigma+\int_S\Big[P\frac{\partial v}{\partial x}-Q\frac{\partial u}{\partial y}+U\left(\frac{\partial v'}{\partial y}-\frac{\partial u}{\partial x}\right)+T\frac{\partial v}{\partial z}-S\frac{\partial u}{\partial z}$$

$$-\omega_3(P+Q)+T\omega_1+S\omega_2\Big]\,dS \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Subtracting the equation (3) from the equation (7), we have

$$\int_S \rho\left(v'\frac{d^2u'}{dt^2}-u'\frac{d^2v'}{dt^2}\right)dS=\int_S\left[\omega_3(P+Q)-T\omega_1-S\omega_2\right]dS \;\ldots(8),$$

and the other analogous results,

$$\int_S \rho\left(w'\frac{d^2v'}{dt^2}-v'\frac{d^2w'}{dt^2}\right)dS=\int_S\left[\omega_1(Q+R)-U\omega_2-T\omega_3\right]dS,$$

$$\int_S \rho\left(u'\frac{d^2w'}{dt^2}-w'\frac{d^2u'}{dt^2}\right)dS=\int_S\left[\omega_2(R+P)-S\omega_3-U\omega_1\right]dS.$$

Let us take S' an element of the space, we shall have, denoting by K_1, K_2, K_3 the components of the moving couple,

$$\left.\begin{aligned} K_1 &= (R+Q)\,\omega_1 - U\omega_2 - T\omega_3 \\ K_2 &= (P+R)\,\omega_2 - S\omega_3 - U\omega_1 \\ K_3 &= (Q+P)\,\omega_3 - T\omega_1 - S\omega_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

and, if we take for axes the direction of the principal stresses,

$$\left.\begin{aligned} K_1 &= (Q+R)\,\omega_1 \\ K_2 &= (R+P)\,\omega_2 \\ K_3 &= (P+Q)\,\omega_3 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

and, if we put $2V = P\,(\omega_2^2+\omega_3^2)+Q\,(\omega_3^2+\omega_1^2)+R\,(\omega_1^2+\omega_2^2)$,

we have $$K_1 = \frac{\partial V}{\partial \omega_1}, \quad K_2 = \frac{\partial V}{\partial \omega_2}, \quad K_3 = \frac{\partial V}{\partial \omega_3}.$$

If $P = Q = R$, we obtain the potential energy of an isotropic solid as given by Sir W. Thomson (*Phil. Mag.*, Vol. XXVI., p. 418).

Note on the G-function in an Elliptic Transformation Annihilator.

By John Griffiths, M.A.

[*Read May 9th*, 1889.]

Notation.— $\partial_k \equiv \dfrac{d}{dk}$; $\partial_x \equiv \dfrac{d}{dx}$, $\Delta x = \sqrt{1-x^2\,.\,1-k^2x^2}$,

$$S^2 = Q^2 - P^2, \quad R^2 = Q^2 - \lambda^2 P^2,$$

where P and Q are functions of x and k.

$$\ldots\ 1 = \lambda^2 + \lambda'^2.$$

[See previous notes by the writer, *Proc. Lond. Math. Soc.*, Vol. XVIII., 1887.]

Section I.

The algebraic theory of the function G is based on the following elementary proposition, viz.:—

If $$\Omega \equiv nkk'^2 \partial_k + \Delta x\, G(x, k)\, \partial_x,$$

$$\frac{M\,dy}{\sqrt{1-y^2 \,.\, 1-\lambda^2 y^2}} = \frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2 x^2}},$$

and $$\Omega\left(\frac{1-y^2}{1-\lambda^2 y^2}\right) = 0,$$

then $$M^{-2}\lambda^2 (1-y^2) = \Delta x \frac{dG}{dx} + \frac{nk^2 k'^2 x^2}{1-k^2 x^2} - \frac{nkk'^2}{M}\frac{dM}{dk};$$

where $$M^{-2} = \frac{nkk'^2}{\lambda\lambda'^2}\frac{d\lambda}{dk},$$

and n is a number.

The proof of this important theorem offers no difficulty. It depends on the identity

$$\frac{d^2 y}{dx\,dk} = \frac{d^2 y}{dk\,dx}.$$

In fact, writing the condition

$$\Omega\left(\frac{1-y^2}{1-\lambda^2 y^2}\right) = 0,$$

in the form $$\frac{\Omega y^2}{1-y^2} = \frac{\Omega \lambda^2 y^2}{1-\lambda^2 y^2} = \frac{y^2 \Omega \lambda^2}{\lambda'^2} = 2M^{-2}\lambda^2 y^2,$$

we have $$nkk'^2 \frac{dy}{dk} + \Delta x \,.\, G \,.\, \frac{dy}{dx} = M^{-2}\lambda^2 y\,(1-y^2),$$

and $$\frac{dy}{dx} = \sqrt{1-y^2 \,.\, 1-\lambda^2 y^2} \div M\Delta x = X, \text{ say}.$$

Hence $$\partial\,\{M^{-2}\lambda^2 y\,(1-y^2) - \Delta x \,.\, G \,.\, X\} = nkk'^2 \frac{dX}{dk},$$

$$\partial_x (\Delta x \,.\, G \,.\, X) + nkk'^2 \frac{dX}{dk} = M^{-2}\lambda^2 (1-3y^2)\, X,$$

$$nkk'^2 \partial_k \log X + \Delta x \,.\, G \,.\, \partial_x \log X + \partial_x (\Delta x \,.\, G) = M^{-2}\lambda^2 (1-3y^2).$$

Now
$$\partial_k \log X = -\frac{1}{M}\frac{dM}{dk} - \left(\frac{y}{1-y^2} + \frac{\lambda^2 y}{1-\lambda^2 y^2}\right)\frac{dy}{dk}$$
$$-\frac{M^{-2}\lambda^2\lambda'^2}{nkk'^2}\frac{y^2}{1-\lambda^2 y^2} + \frac{kx^2}{1-k^2x^2},$$
$$\partial_x \log X = \frac{x}{1-x^2} + \frac{k^2 x}{1-k^2x^2} - \left(\frac{y}{1-y^2} + \frac{\lambda^2 y}{1-\lambda^2 y^2}\right)\frac{dy}{dx},$$
so that
$$-\frac{nkk'^2}{M}\frac{dM}{dk} - \frac{M^{-2}\lambda^2\lambda'^2 y^2}{1-\lambda^2y^2} + \frac{nk^2k'^2x^2}{1-k^2x^2} + G\,.\,\partial_x\,\Delta x + \Delta x\,.\,\partial_x G$$
$$-\left(\frac{y}{1-y^2} + \frac{\lambda^2 y}{1-\lambda^2y^2}\right)\left(nkk'^2\frac{dy}{dk} + \Delta x\,.\,G\,.\,\frac{dy}{dx}\right)$$
$$+\left(\frac{x}{1-x^2} + \frac{k^2x}{1-k^2x^2}\right)\Delta x\,.\,G = M^{-2}\lambda^2(1-3y^2),$$
i.e.,
$$-\frac{nkk'^2}{M}\frac{dM}{dk} - \frac{M^{-2}\lambda^2\lambda'^2y^2}{1-\lambda^2y^2} - M^{-2}\lambda^2y^2 - M^{-2}\lambda^4\frac{y^2(1-y^2)}{1-\lambda^2y^2}$$
$$+\frac{nk^2k'^2x^2}{1-k^2x^2} + G\,.\,\partial_x\,\Delta x + \Delta x\,\partial_x G + \left(\frac{x}{1-x^2} + \frac{k^2x}{1-k^2x^2}\right)\Delta x\,.\,G$$
$$= M^{-2}\lambda^2(1-3y^2),$$
or finally, since $\partial_x\,\Delta x + \left(\frac{x}{1-x^2} + \frac{k^2x}{1-k^2x^2}\right)\Delta x = 0,$

and
$$\lambda'^2 + \lambda^2(1-y^2) = 1-\lambda^2y^2,$$
we have
$$\Delta x\frac{dG}{dx} + \frac{nk^2k'^2x^2}{1-k^2x^2} - \frac{nkk'^2}{M}\frac{dM}{dk} = M^{-2}\lambda^2(1-y^2).$$

Section 2.

Differential equations satisfied by the function $G(x, k)$.

From the above it appears that $G(x, k)$ satisfies the following differential equations of the second order, viz.:

$$\frac{\pm\, d\xi}{\sqrt{\xi\,.\,M^{-2}\lambda^2-\xi\,.\,M^{-2}\lambda'^2+\xi}} = \frac{2\,dx}{\sqrt{1-x^2\,.\,1-k^2x^2}} \quad\ldots\ldots\ldots\ldots(1),$$

$$\Omega\xi = 2\xi\left\{\xi - \frac{nkk'^2}{M}\frac{dM}{dk} + M^{-2}(1-2\lambda^2)\right\} \quad\ldots\ldots\ldots\ldots\ldots(2),$$

since
$$\xi = \Delta x\frac{dG}{dx} + \frac{nk^2k'^2x^2}{1-k^2x^2} - \frac{nkk'^2}{M}\frac{dM}{dk} = M^{-2}\lambda^2(1-y^2),$$

and $$\Omega\left\{\frac{\xi}{\lambda^2(\xi+M^{-2}\lambda'^2)}\right\}=0,$$

where $$\Omega\equiv nkk'^2\partial_k+\Delta x\,.\,G\partial_x.$$

Section 3.

Problem of Elliptic Transformation.

It is evident from the preceding sections that the function G is of very great importance with regard to elliptic transformation. Practically the only case to be considered is the following, viz., when

$$y=\frac{P(x,k)}{Q(x,k)}\text{ gives }\frac{M\,dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}}=\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}},$$

$$M^{-2}=\frac{nkk'^2}{\lambda\lambda'^2}\,\frac{d\lambda}{dk},$$

and G has the four values,

$$1.\quad G=\Delta x\partial_x\log\frac{P}{(1-k^2x^2)^{\frac{1}{2}n}},$$

$$2.\quad G=\Delta x\partial_x\log\frac{Q}{(1-k^2x^2)^{\frac{1}{2}n}},$$

$$3.\quad G=\Delta x\partial_x\log\frac{S}{(1-k^2x^2)^{\frac{1}{2}n}},$$

$$4.\quad G=\Delta x\partial_x\log\frac{R}{(1-k^2x^2)^{\frac{1}{2}n}}.\quad\text{(See notation).}$$

Corresponding to these four values, we have the formulæ—

1. $$\Omega\equiv nkk'^2\partial_k+\Delta x\,.\,G\,.\,\partial_x,$$

$$\Omega\,\frac{\lambda S}{R}=0,\qquad G=\Delta x\partial_x\log\frac{P}{(1-k^2x^2)^{\frac{1}{2}n}},$$

$$M^{-2}\lambda^2\left(1-\frac{Q^2}{\lambda^2P^2}\right)=\Delta x\frac{dG}{dx}+\frac{nk^2k'^2x^2}{1-k^2x^2}-\frac{nkk'^2}{M}\,\frac{dM}{dk},$$

$$M^{-2}Q^2=(C+nk^2x^2)\,P^2+(\Delta x)^2\,(P'^2-PP'')+x\,(1+k^2-2k^2x^2)\,PP',$$

where $$C=M^{-2}\lambda^2+\frac{nkk'^2}{M}\,\frac{dM}{dk}-nk^2,$$

$$P'\equiv\frac{dP}{dx},\qquad P''=\frac{d^2P}{dx^2}.$$

2. $$\Omega \equiv nkk'^2\partial_k + \Delta x \,.\, G \,.\, \partial_x,$$

$$\Omega\,\frac{S}{R} = 0\,; \quad G = \Delta x\,\partial_x \log\frac{Q}{(1-k^2x^2)^{\frac{1}{2}n}},$$

$$M^{-2}\lambda^2\left(1-\frac{P^2}{Q^2}\right) = \Delta x\,\frac{dG}{dx} + \frac{nk^2k'^2x^2}{1-k^2x^2} - \frac{nkk'^2}{M}\,\frac{dM}{dk},$$

$$M^{-2}\lambda^2 P = (C+nk^2x^2)\,Q^2 + (\Delta x)^2\,(Q'^2 - QQ'') + x\,(1+k^2-2k^2x^2)\,QQ',$$

where $$C = M^{-2}\lambda^2 + \frac{nkk'^2}{M}\,\frac{dM}{dk} - nk^2,$$

$$Q' \equiv \frac{dQ}{dx}, \quad Q'' = \frac{d^2Q}{dx^2}.$$

3. $$\Omega\lambda\,\frac{P}{Q} = 0, \qquad G = \Delta x\,\partial_x \log\frac{S}{(1-k^2x^2)^{\frac{1}{2}n}},$$

$$M^{-2}R^2 = (C+nk^2x^2)\,S^2 + (\Delta x)^2\,(S'^2 - SS'') + x\,(1+k^2-2k^2x^2)\,SS'.$$

4. $$\Omega\,\frac{P}{Q} = 0, \qquad G = \Delta x\,\partial_x \log\frac{R}{(1-k^2x^2)^{\frac{1}{2}n}},$$

$$M^{-2}\lambda^2 S^2 = (C+nk^2x^2)\,R^2 + (\Delta x)^2\,(R'^2 - RR'') + x\,(1+k^2-2k^2x^2)\,RR',$$

where C has the same value as before, and

$$R' \equiv \frac{dR}{dx}, \quad \&c.$$

If we take $$P = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,$$
$$Q = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n,$$

the conclusion is that the problem of finding the rational transformation equations $y = \dfrac{P}{Q}$ can be reduced to the following one, viz., to determine the k-functions $a_0, a_1, \ldots a_n, b_0, b_1, \ldots b_n$, so that the expressions

$$(C+nk^2x^2)\,P^2 + (\Delta x)^2\,(P'^2 - PP'') + x\,(1+k^2-2k^2x^2)\,PP',$$
$$(C+nk^2x^2)\,Q^2 + (\Delta x)^2\,(Q'^2 - QQ'') + x\,(1+k^2-2k^2x^2)\,QQ'$$

shall be each a square, *i.e.*, $M^{-2}Q^2$ and $M^{-2}\lambda^2P^2$, respectively. The value of the function C is

$$M^{-2}\lambda^2 - nk^2 + \frac{nkk'^2}{M}\,\frac{dM}{dk},$$

i.e., since $$M^{-2} = \frac{nkk'^2}{\lambda\lambda'^2}\,\frac{d\lambda}{dk},$$

we have $$C = \frac{n}{2}\, kk'^2 \left(\frac{\dot\lambda}{\lambda} - \frac{\ddot\lambda}{\dot\lambda}\right) - \frac{n}{2}\, k'^2,$$

if $$\dot\lambda \equiv \frac{d\lambda}{dk}, \quad \ddot\lambda = \frac{d^2\lambda}{dk^2}.$$

Section 4.

An example of the above method of squares.

If $$P = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

it is proved without difficulty that the expression

$$(C + nk^2x^2)\, P^2 + (\Delta x)^2\, (P'^2 - PP'') + x\,(1 + k^2 - 2k^2x^2)\, PP'$$

is a rational and integral function as regards x of a degree not higher than $2n$ since the coefficients of x^{2n+1} and x^{2n+2} vanish. It may happen, of course, that the degree in question is lower than $2n$.

I consider an example of the method of squares in order to illustrate the above formulæ.

Let $$P = a_1 x + a_3 x^3, \quad Q = 1 + b_2 x^2,$$

$n = 3$, so that

$$Q' = 2b_2 x, \quad Q'' = 2b_2, \quad Q'^2 - QQ'' = b_2^2 x^2 - 2b_2,$$

$$C = M^{-2}\lambda^2 - 3k^2 + \frac{3kk'^2}{M}\,\frac{dM}{dk} = \tfrac{3}{2} kk'^2 \left(\frac{\dot\lambda}{\lambda} - \frac{\ddot\lambda}{\dot\lambda}\right) - \tfrac{3}{2} k'^2,$$

$$M^{-2}\lambda^2 P'^2 = (C + 3k^2x^2)(1 + b_2x^2)^2 + 2\,(1 - x^2)(1 - k^2x^2)(b_2^2x^2 - b_2)$$
$$+ 2b_2x^2\,(1 + k^2 - 2k^2x^2)(1 + b_2x^2),$$

i.e., $$M^{-2}\lambda^2 P'^2 = C - 2b_2 + \left\{6b_2^2 + 4\,(1 + k^2)\, b_2 + 3k^2\right\} x^2 + 2b_2^3 x^4 + k^2 b_2^2 x^6.$$

Hence it is seen at once that this expression will be a square if

$$C = 2b_2 \quad\text{and}\quad b_2^4 = k^2\left\{6b_2^2 + 4\,(1 + k^2)\, b_2 + 3k^2\right\}.$$

We have thus b_2 given as a k-function, and

$$M^{-1}\lambda P = x\left(\frac{b_2^2}{k} + b_2 k x^2\right) \quad\text{or}\quad P = M^{-1}x\left(1 - \frac{x^2}{a^2}\right),$$

if $$b_2^2 = M^{-2}\lambda k, \quad b_2 k = -\frac{M^{-2}\lambda}{a^2},$$

and, consequently,

$$b_2 = -k^2a^2, \quad k^4a^8-6k^2a^4+4(1+k^2)a^2-3 = 0.$$

Similarly, taking $$P = M^{-1}x\left(1-\frac{x^2}{a^2}\right),$$

the expression for Q^2 is

$$Q^2 = (2b_2+3k^2x^2)\,x^2\left(1-\frac{x^2}{a^2}\right)^2+(\Delta x)^2\left(1+3\frac{x^4}{a^4}\right)$$
$$+x^2(1+k^2-2k^2x^2)\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{3x^2}{a^2}\right),$$

$$(1+b_2x^2)^2 = 1+2b_2x^2+\left\{6k^2-4\frac{1+k^2}{a^2}+\frac{3}{a^4}\right\}x^4,$$

so that $$6k^2-4\frac{1+k^2}{a^2}+\frac{3}{a^4} = b_2^2 = k^4a^4,$$

i.e., $$k^4a^8-6k^2a^4+4(1+k^2)a^2-3 = 0, \text{ as before.}$$

The modular relation between λ and k is obtained from two differential equations, viz.,

$$\frac{3k^2k'^2}{\lambda'^2}\dot{\lambda} = M^{-2}\lambda k = b_2^2 \text{..........................}(1),$$

$$C = 2b_2, \text{ or } kk'^2\left\{\frac{\dot{\lambda}}{\lambda}-\frac{\ddot{\lambda}}{\dot{\lambda}}\right\}-k'^2 = \tfrac{4}{3}b_2 \text{.. ....}(2),$$

where $$\dot{\lambda} \equiv \frac{d\lambda}{dk}, \quad \ddot{\lambda} \equiv \frac{d^2\lambda}{dk^2},$$

and $$b_2^4-6k^2b_2^2-4k^2(1+k^2)\,b_2-3k^4 = 0.$$

By means of these equations we may eliminate $\dot{\lambda}$ and $\ddot{\lambda}$, and so determine λ in terms of k and the k-function b_2. The results are simplified by writing $\lambda k = l^2$,

$$b_2 = \frac{l-k^2}{1-l}, \quad l^4-4k^2l^3+6k^2l^2-4k^2l+k^4 = 0,$$

$$k\frac{\dot{\lambda}}{\lambda} = \frac{(l-k^2)^2(1-l)^2}{3k'^4l^2}, \quad M^{-1} = \frac{k^2-l}{l(1-l)}.$$

[If $\lambda = k$, we have

$$\dot{\lambda} = 1, \quad \ddot{\lambda} = 0, \quad M^{-2} = \frac{nkk'^2}{\lambda\lambda'^2}\dot{\lambda} = n,$$

$$C = \frac{2}{n} kk'^2 \left\{ \frac{\dot{\lambda}}{\lambda} - \frac{\ddot{\lambda}}{\dot{\lambda}} \right\} - \frac{n}{2} k'^2 = 0.$$

This case is possible when n is a square number.]

SECTION 5.

Differential equation satisfied by the numerator and denominator P, Q of a transformation equation $y = \dfrac{P(x, k)}{Q(x, k)}$.

From Section 3 it appears that we have

$$M^{-2}\lambda^2 P^2 = (C + nk^2x^2)\, Q^2 + (\Delta x)^2 (Q'^2 - QQ'') + x(1 + k^2 - 2k^2x^2)\, QQ',$$

$$M^{-2} Q^2 = (C + nk^2x^2)\, P^2 + (\Delta x)^2 (P'^2 - PP'') + x(1 + k^2 - 2k^2x^2)\, PP',$$

where
$$C = M^{-2}\lambda^2 + \frac{nkk'^2}{M}\frac{dM}{dk} - nk^2,$$

$$Q' \equiv \frac{dQ}{dx}, \quad Q'' = \frac{d^2Q}{dx^2}, \text{ \&c.}$$

Hence, with the above notation, if we write

$$f(x) = (C + nk^2x^2)\, Q^2 + (\Delta x)^2 (Q'^2 - QQ'') + x(1 + k^2 - 2k^2x^2)\, QQ',$$

then Q satisfies the differential equation of the fourth order

$$2M^{-4}\lambda^2 f(x)\, Q^2 = 2(C + nk^2x^2)\{f(x)\}^2 + (\Delta x)^2\{[f'(x)]^2 - f(x) f''(x)\}$$
$$+ x(1 + k^2 - 2k^2x^2) f(x) f'(x).$$

Similarly P, S, and R satisfy the same equation.

SECTION 6.

Jacobi's partial differential equation—satisfied by the numerator and denominator P and Q.

It remains to point out how Jacobi's equation can be deduced from the above annihilator theory of elliptic transformation.

Now, if
$$\Omega \equiv nkk'^2 \partial_k + \Delta x \,.\, G\partial_x,$$
$$\xi = \Delta x \frac{dG}{dx} + \frac{nk^2k'^2x^2}{1-k^2x^2} - \frac{nkk'^2}{M}\frac{dM}{dk},$$

then
$$\Omega\xi = 2\xi \left\{ \xi - \frac{nkk'^2}{M}\frac{dM}{dk} + M^{-2}(1-2\lambda^2) \right\} \quad \text{(see Sect. 2).}$$

Putting
$$G = \Delta x\, \partial_x \log \frac{Q}{(1-k^2x^2)^{\frac{1}{2}n}},$$

we have
$$\xi = nk^2(1-x^2) - x(1+k^2-2k^2x^2)\frac{Q'}{Q} + (\Delta x)^2\left(\frac{Q''}{Q} - \frac{Q'^2}{Q^2}\right) - \frac{nkk'^2}{M}\frac{dM}{dk},$$
$$\Omega \equiv nkk'^2\partial_k + \left\{ nk^2x(1-x^2) + (\Delta x)^2 \frac{Q'}{Q} \right\} \partial_x,$$

so that the equation $\Omega\xi = 2\xi\,\{\xi + \text{a } k\text{-function}\}$

ultimately gives the result
$$X_1Q^3 + X_2Q^2 + X_3Q + X_4 = 0,$$

where X_1, X_2, X_3, X_4 are functions of x, $\frac{dQ}{dx}$, $\frac{dQ}{dk}$, $\frac{d^2Q}{dx^2}$, &c.

Practically we have only to determine the function X_4. This is
$$X_4 = (\Delta x)^2 \left\{ (\Delta x)^2 \frac{d^2Q}{dx^2} + xx_1\frac{dQ}{dx} + 2nkk'^2\frac{dQ}{dk} \right\} \left(\frac{dQ}{dx}\right)^2,$$

where
$$x_1 = (2n-1)\,k^2 - 1 - 2\,(n-1)\,k^2x^2.$$

Hence it is easily seen that, if Q contains neither of the expressions $1-x^2$, $1-k^2x^2$, nor a square factor, the function
$$(\Delta x)^2\frac{d^2Q}{dx^2} + xx_1\frac{dQ}{dx} + 2nkk'^2\frac{dQ}{dk}$$

must have Q for a factor, supposing Q to be a rational and integral function of x of the form
$$Q = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n.$$

We, therefore, have finally
$$(\Delta x)^2\frac{d^2Q}{dx^2} + xx_1\frac{dQ}{dx} + 2nkk'^2\frac{dQ}{dk} = (\alpha + \beta x + \gamma x^2)\,Q,$$

where α, β, and γ are constants independent of x.

By comparing coefficients we easily find that in general $\beta = 0$ and $\gamma = -n(n-1)k^2$, so that

$$(\Delta x)_2 \frac{d^2Q}{dx^2} + xx_1 \frac{dQ}{dx} + 2nkk'^2 \frac{dQ}{dk} = \{a - n(n-1)k^2x^2\} Q.$$

It is beyond the scope of the present note to discuss the results which flow from the above equation. I merely write down the formula which connects the coefficients of Q, viz.,

If $$Q = a_0 + a_1 x + a_2 x^2 + \dots + a_t x^t + \dots + a_n x^n,$$

and $a_0 = 1$, then

$$(t+4)(t+3)\, a_{t+4} - \{2a_2 + (t+2)^2 + (t+2)(t+2-2n)\, k^2\}\, a_{t+2}$$
$$+ 2nkk'^2 \dot{a}_{t+} + (n-t)(n-t-1)\, k^2 a_t = 0,$$

where $$\dot{a} \equiv \frac{da}{dk}.$$

For example, in Jacobi's transformation equations, we have

$$Q = 1 + a_2 x^2 + a_4 x^4 + \dots + a_{n-1} x^{n-1} \quad (n \text{ an odd number}),$$

and the formula shows that, if a_2 be a known k-function, then all the remaining coefficients $a_4, a_6 \dots a_{n-1}$ are known.

In this instance we also see that a_2 satisfies a differential equation of the order $\frac{1}{2}(n-1)$.

The case of complete multiplication by $\sqrt{n}$ when n is a square number, is particularly interesting and simple, since then we have $a_2 = 0$.

Section 7.

Transformation of the Function $\Theta(u)$.

The transformation of this function follows immediately from the theory of the G-function. See Note by the present writer (*Proc. Lond. Math. Soc.*, Vol. xviii., 1887, p. 383).

On Cyclotomic Functions. By H. W. Lloyd Tanner, M.A., Professor of Mathematics in the University College of South Wales.

SECTION III.

The cyclctomics which belong to the f-nomial periods of the p^{th} roots of unity, when p is a prime number.

[*Read May 9th*, 1889.]

Abstract. (Arts. 1–7.)

1. When p is an odd prime, $= 2e+1$ say, the binomial periods of the p^{th} roots of unity are

$$x+x^{-1},\quad x^2+x^{-2},\ \dots\ x^e+x^{-e},$$

where x is one of the roots of

$$x^{p-1}+x^{p-2}+\dots+x+1=0.$$

The periods are the roots of the well-known equation

$$\eta^e+\eta^{e-1}-(e-1)\,\eta^{e-2}-(e-2)\,\eta^{e-3}$$
$$+\frac{(e-2)(e-3)}{1\,.\,2}\,\eta^{e-4}+\frac{(e-3)(e-4)}{1\,.\,2}\,\eta^{e-5}+\dots=0,$$

the expression on the left being continued to $e+1$ terms. The object of the present communication is to give the corresponding theorem for f-nomial periods of the p^{th} roots of unity when $p, = ef+1$, is a prime number. The difficulty is, that we have not for the f-nomial periods, as we have for the binomial periods, a form which can be written down without knowing p. For example, the leading trinomial periods of the 7^{th}, 13^{th}, 19^{th} roots of unity are

$$x+x^2+x^{-3},\quad x+x^3+x^{-4},\quad x+x^7+x^{-8},$$

respectively; and there is no general expression for these indices analogous to the expression ± 1 for the indices of the leading binomial period.

2. The result obtained is that the cyclotomic may be regarded as a product of three "factors." Each of these consists of an infinite number of terms with integral coefficients, the first coefficient being 1. The cyclotomic consists of the first $e+1$ terms of the product, and the remaining terms of the product are zeros as far as the term containing η^{-n}, if each of the three factors is calculated to this extent.

3. One of the factors, called the *asymptotic factor*, is the only one that appears in the binomial-period equation written above. For a given f, it is a series depending on e, and appropriate to every value of p, $= ef+1$. When f is a prime, the first f coefficients are independent of e; they are in fact identical with the first f coefficients of the expansion of $(1-fy)^{-1/f}$. The f coefficients which follow are linear functions of e, or more conveniently of ϵ, $= (f-1)!\,e$. In these linear functions the ϵ occurs multiplied by the first f coefficients of the asymptotic factor. The following sets of f coefficients are quadric, cubic, &c. functions of ϵ. For examples, see the tables for $f = 3, 5, 7$, appended to this paper.

When f is composite, each factor of f affects the form of the asymptotic factor; but the smallest factor has the most obvious influence. For instance, when f is even, the factor proceeds in pairs of terms, like the binomial period cyclotomic. Examples will be found in the tables for $f = 4, 6, 8, 9$.

The asymptotic factor presents itself as a product of other factors, one of which is $(1-fx)^{-1/f}$, and the others are "central" factors. The coefficients of these subsidiary factors are not all integral.

4. The second factor of the cyclotomic—called the *eccentric factor* —is not expressed in terms of e; so that it has to be calculated separately for each value of p. All its coefficients except the first are multiples of p; that is, it is of the form

$$1-p\,\{E_k y^k + E_{k+1} y^{k+1} + \dots\}, \quad (y = \eta^{-1}),$$

where E_k, E_{k+1} are integers, and are positive at least as far as E_{2k}. This, like the asymptotic factor, naturally splits up into a product of other factors; but, unlike the asymptotic factor, all its factors have integral coefficients.

From the form of the eccentric factor it follows that the asymptotic factor is congruent to the cyclotomic, mod. p. There is a presumption in favour of the theorem, that by taking p sufficiently large, k may be made as large as we please: a theorem which would justify the epithet "asymptotic." But this is not proved. On the assumption that certain forms (for instance, the geometric series $1+a+\dots+a^{f-1}$ when f is prime) contain an infinite number of primes, the theorem can be proved; and, as the case of f prime requires very little space, I have discussed it. It did not seem worth while to extend the discussion: not only because it was based upon an unproved, though probable, assumption; but also because the inferior limit of k, determined in this way, seems to hold good for all values of p, and not merely for those values of p implied in the assumption.

5. To calculate the cyclotomic, it is sufficient to take the product of the asymptotic and eccentric factors. This product, however, differs from the cyclotomic. For instance, $e=3$, $e=2$, $p=7$; the coefficients of the product of the asymptotic and eccentric factors are

$$1,\ 1,\ 2,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 2,\ 0,\ 0,\ \dots\dots$$

and to make this agree with cyclotomic

$$1,\ 1,\ 2,$$

a *third factor*, $\qquad 1-\eta^{-p}+A\eta^{-2p}+\&c.,$

must be introduced. Although this factor is absolutely without influence on the calculation of the cyclotomic, yet it seems satisfactory to explain how such a factor arises: and this is done in the sequel.

The expression used for forming the cyclotomic for binomial periods is the asymptotic factor only. The eccentric factor in this case is $1+py^{e-2}+\dots$, and the third factor is $1-py^{e}-\&c.$; so that neither of these influences the coefficients that are to be determined.

6. The arrangement of the work will now be indicated. The expression for log $\mathfrak{P}$, $\mathfrak{P}$ being the cyclotomic, is first formed. In the analysis of this expression, considerable use is made of a regular polygon of f sides, at the vertices of which are placed particles of various weights, all commensurable. According as the centre of gravity of these particles is or is not at the centre of the polygon, the system is termed a central or an eccentric system. The central factors of $\mathfrak{P}$ come from the central systems; and the eccentric factors from the eccentric systems. This weighted polygon promises to be useful in discussing complex numbers formed with roots of unity; but the application is hardly within the scope of the present communication.

7. It was necessary to find the conditions that a series

$$-s_1y-s_2y^2/2-s_3y^3/3-\&c.$$

should be the logarithm of a series

$$1+P_1y+P_2y^2+\dots$$

with integral coefficients. These conditions are obtained in the form

$$s_n+\Sigma s_\delta-\Sigma s_\epsilon\equiv 0,\ \text{mod. } n,$$

where δ, ϵ are divisors of n, such that n/δ is a product of an even number of primes all different; and n/ϵ is a product of an odd num-

ber of primes all different. For instance,

$$s_6 - s_3 - s_2 + s_1 \equiv 0, \text{ mod. } 6,$$

$$s_{12} - s_6 - s_4 + s_2 \equiv 0, \text{ mod. } 12.$$

To prove this, the transformation

$$1 + P_1 y + P_2 y^2 + \ldots = (1 - Q_1 y)(1 - Q_2 y^2)(1 - Q_3 y^3)\ldots$$

is employed. The use of the coefficients Q turns out to be very labour-saving in passing to a series from its logarithm; and these coefficients appear to be of some significance in other respects. For instance, in the expansion of

$$(1 - fy)^{-1/f},$$

all the Q whose subscripts are prime to f are integral, and these Q remain unchanged in the asymptotic factor of the cyclotomic.

It will be seen that, though the object of this paper was to consider especially the case of a determinate f, yet some results have a bearing on the question of the e-section of a cyclotomic, when f is not determinate. But the paper had extended to such a length that it seemed discreet to postpone the development of this side.

Formation of log $\mathfrak{P}$. (Arts. 8–13.)

8. Let x be a root of the equation

$$x^{p-1} + x^{p-2} + \ldots + x + 1 = 0,$$

where p is a prime number. Writing, as usual,

$$ef = p - 1,$$

where e, f are integers, there is one set of f-nomial periods of x. The set consists of e periods of which the leading period is

$$\eta = x + x^a + x^{a^2} + \ldots + x^{a^{f-1}},$$

where a is a root of unity (mod. p) of order f, that is to say,

$$a^f \equiv 1, \text{ mod. } p,$$

but no lower power of a is congruent to unity.

These e periods are the roots of an equation

$$\eta^e + P_1 \eta^{e-1} + P_2 \eta^{e-2} + \ldots + P_{e-1} \eta + P_e = 0,$$

where the coefficients P are integers. The expression on the left is the cyclotomic function discussed in this section. It is convenient to divide throughout by η^e, and then write y for η^{-1}, so that the cyclo-

tomic becomes

$$\mathfrak{Y} = 1 + P_1 y + P_2 y^2 + \ldots + P_e y^e.$$

To determine the coefficients, P, we make use of the logarithm of the cyclotomic, viz.,

$$\log \mathfrak{Y} = -s_1 y - s_2 \frac{y^2}{2} - s_3 \frac{y^3}{3} - \ldots.$$

The series on the right is infinite, and the s are the power sums of the e periods.

9. Since $$\eta = x + x^a + x^{a^2} + \ldots + x^{a^{f-1}},$$

we have $$\eta^\sigma = (x + x^a + x^{a^2} + \ldots + x^{a^{f-1}})^\sigma$$
$$= A_\sigma + B_\sigma x + C_\sigma x^2 + \ldots + D_\sigma x^{p-1},$$

where A_σ represents the sum of the coefficients of x^p, x^{2p}, x^{3p}, &c. in the expansion of η^σ; B_σ means the sum of the coefficients of x, x^{p+1}, x^{2p+1}, &c. in the same expansion: and the like for the other letters. This transformation only postulates $x^p = 1$; and we may therefore put $x = 1$, which gives

$$A_\sigma + B_\sigma + C_\sigma + \ldots + D_\sigma = f^\sigma.$$

10. The expression for s_σ, the sum of the σ^{th} powers of the e periods, is at once formed from the value of η^σ, by taking account of the fact that s_σ is a symmetrical function of $x, x^2, \ldots x^{p-1}$. We have, namely,

$$s_\sigma = eA_\sigma + (B_\sigma + C_\sigma + \ldots + D_\sigma)(x + x^2 + \ldots + x^{p-1}) \frac{e}{p-1}$$
$$= eA_\sigma - (B_\sigma + C_\sigma + \ldots + D_\sigma) \frac{1}{f}$$

(since $x + x^2 + \ldots + x^{p-1} = -1$ and $p - 1 = ef$)

$$= \left(e + \frac{1}{f}\right) A_\sigma - (A_\sigma + B_\sigma + C_\sigma + \ldots + D_\sigma) \frac{1}{f}$$

$$= \frac{p}{f} A_\sigma - f^{\sigma-1}.$$

11. From this we obtain

$$\log \mathfrak{Y} = \Sigma f^{\sigma-1} \frac{y^\sigma}{\sigma} - \frac{p}{f} \Sigma A_\sigma \frac{y^\sigma}{\sigma},$$

the summations extending to all positive integral values of σ.

The first sum may be written

$$\frac{1}{f}\,\Sigma f^{\sigma}y^{\sigma}/\sigma = -\frac{1}{f}\log(1-fy),$$

so that

$$(1-fy)^{-1/f}$$

is a "factor" of $\mathfrak{P}$.

12. To determine the value of A_{σ}, observe that it is the sum of the coefficients of x^{p}, x^{2p}, ... in the expansion of η^{σ}, so that

$$A_{\sigma} = \Sigma\frac{\sigma}{\lambda_0!\,\lambda_1!\,\ldots\,\lambda_{f-1}!},$$

where

$$\lambda_0+\lambda_1+\ldots+\lambda_{f-1} = \sigma,$$

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-1}a^{f-1} \equiv 0, \text{ mod. } p,$$

and the summation includes every system of positive integers,

$$\lambda_0, \lambda_1, \ldots \lambda_{f-1},$$

which satisfy this double condition.

13. Considering two systems

$$(\lambda_0, \lambda_1, \ldots) \text{ and } (\mu_0, \mu_1, \ldots),$$

it is obvious that, if each element of one system is equal to the corresponding element in the other, the two systems give the same term in A_{σ}; that is, in calculating A_{σ}, only one of them is counted. But, if any one of the equations $\lambda_0=\mu_0$, $\lambda_1=\mu_1$, ... is not satisfied, the two systems give different terms in A_{σ}, and each system contributes its full quota to A_{σ}.

14. It is clear that the congruence is the only effective condition, for the equation may be regarded as merely determining the rank of the A to which a system $(\lambda_0, \lambda_1, \ldots)$ contributes. Accordingly, a considerable part of the sequel relates to the theory of the solutions of the congruence. We proceed to classify the solutions, *firstly* into recurring and non-recurring systems (Arts. 15, 16), and *secondly* into central and eccentric systems (Arts. 17–42).

Recurring and non-recurring systems. (Arts. 15, 16.)

15. From any system $\lambda_0, \lambda_1, \ldots \lambda_{f-1}$,

which satisfies the congruence

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-1}a^{f-1} \equiv 0, \text{ mod. } p,$$

we can by cyclic substitution derive others which also satisfy the congruence. For, whatever integer k may be,

$$\lambda_k,\ \lambda_{k+1},\ \dots\ \lambda_{f-1},\ \lambda_0,\ \lambda_1,\ \dots\ \lambda_{k-1}$$

is a solution, as is seen on multiplying both sides of the congruence into a^{f-k}. Every derived system gives to A_σ the same amount as the original system, so that, if the derived systems are distinct from each other, and from the original system (*cf.* Art. 13), the complete contribution from A_σ is

$$f\,\frac{\sigma!}{\lambda_0!\,\lambda_1!\,\dots\,\lambda_{f-1}!}.$$

On reference to the value of $\log \mathfrak{P}$ it will be seen that the denominator of the fractional multiplier cancels out.

16. It may, however, happen that the systems obtained by cyclic substitution are not all different; say

$$(\lambda_0,\ \lambda_1,\ \dots\ \lambda_h\ \dots) = (\lambda_h,\ \lambda_{h+1},\ \dots\ \lambda_{2h},\ \dots).$$

This means that every member of the first system is equal to the corresponding member of the second; that is to say,

$$\lambda_0 = \lambda_h = \lambda_{2h} = \dots,\quad \lambda_1 = \lambda_{h+1} = \lambda_{2h+1} = \dots,\ \&c.$$

The original system may therefore be written

$$\lambda_0,\ \lambda_1,\ \dots\ \lambda_{h-1},\ \lambda_0,\ \lambda_1\ \dots\ \lambda_{h-1},\ \lambda_0\ \dots\dots\ \lambda_{h-1},$$

and consists of f/h $(= g)$ cycles. Of the f systems only h are distinct from each other; viz., these are the systems which begin with $\lambda_0, \lambda_1, \lambda_2 \dots \lambda_{h-1}$ respectively. The contribution of the whole set to $\log \mathfrak{P}$ is therefore

$$\frac{p}{f}\,h\,\frac{\sigma!}{(\lambda_0!\,\dots\,\lambda_{h-1}!)^g} = \frac{p}{g}\,\frac{\sigma!}{(\lambda_0!\,\dots\,\lambda_{h-1}!)^g}.$$

Herein g is any divisor of f, including 1 and f. When $g = 1$, the system consists of a single cycle; in other words, it is a non-recurring system, as in Art. 15. For every other value of g the system is recurring, and the total contribution to $\log \mathfrak{P}$ is a fractional multiple of the multinomial coefficient.

Solutions independent of p. (Arts. 17, 18.)

17. A second classification of the systems

$$\lambda_0,\ \lambda_1,\ \dots\ \lambda_{f-1},$$

which satisfy the congruence

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-1}\, a^{f-1} \equiv 0, \text{ mod. } p,$$

depends upon a more important property. There are some systems which are solutions of the congruence for all values of p of the form $ef+1$; while other systems are solutions for some only of these values.

For the quantity a which appears in this congruence, being a primitive f^{th} root of unity to modulus p, satisfies a congruence of degree τf,* say

$$Fa \equiv 0, \text{ mod. } p.$$

If, then, $$\lambda_0+\lambda_1 a+\ldots+\lambda_{f-1}\, a^{f-1}$$

is a multiple of Fa, it is divisible by p.

Now the *equation* $$Fa = 0$$

determines the primitive f^{th} roots of unity, which have nothing to do with p, so that the coefficients of Fa are also independent of p.

It follows that, if

$$\lambda_0+\lambda_1 a+\ldots+\lambda_{f-1}\, a^{f-1} = (\mu_0+\mu_1 a+\ldots)\, Fa,$$

where μ_0, μ_1 are any integers, then

$$\lambda_0,\ \lambda_1,\ \ldots\ \lambda_{f-1}$$

is a solution of the proposed congruence for all values of p of the form $ef+1$.

18. A more useful form of this result consists in the explicit statement of the relations between the λ which are necessary and sufficient to ensure that

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-1}\, a^{f-1}$$

should be a multiple of Fa. If we make use of a function, Ga, defined by the equation

$$Fa\,.\,Ga = 1-a^f,$$

the conditions may be written

$$\lambda^k\,.\,G\lambda = 0,$$

where $k = 0, 1, 2, \ldots, \tau f-1$; and every power λ^i or λ^{f+i} is to be replaced by λ_i.

* τf is Prof. Sylvester's symbol for the totient of f, that is to say, the number of numbers not greater than f and prime to f.

We have, in fact,

$$(\lambda_0+\lambda_1 a+\ldots)\,/\,Fa=(\lambda_0+\lambda_1 a+\ldots)\;Ga\,/\,(1-a^f).$$

Now, let $$Ga=g_0+g_1a+g_2a^2+\ldots+g_{f-1}a^{f-1},$$

and $$(\lambda_0+\lambda_1 a+\ldots)\;Ga=h_0+h_1a+h_2a^2+\ldots.$$

This last expression is divisible by $1-a^f$, if for all values of k

$$h_k+h_{f+k}=0.$$

But $$h_k=g_0\lambda_k+g_1\lambda_{k-1}+\ldots+g_k\lambda_0,$$

and $$h_{f+k}=g_{k+1}\lambda_{f-1}+g_{k+2}\lambda_{f-2}+\ldots+g_{f-1}\lambda_{k+1}.$$

Hence the condition of divisibility is

$$g_0\lambda_k+g_1\lambda_{k-1}+\ldots+g_k\lambda_0+g_{k+1}\lambda_{f-1}+\ldots+g_{f-1}\lambda_{k+1}=0,$$

or, symbolically, $$\lambda^k.\,G\,(\lambda^{-1})=0.$$

Since $$Ga=-a^{-\gamma}.\,Ga^{-1},$$

where γ, $=f-rf$, is such as to make both sides of the same degree in a, the conditions may be written

$$\lambda^k.\,G\lambda=0.$$

Graphic representation. (Arts. 19, 20.)

19. It is convenient to present the matter graphically. At the vertices of a regular convex polygon of f sides, suppose particles to be placed whose weights are $\lambda_0, \lambda_1, \ldots \lambda_{f-1}$. Since the λ represent integers, the weights of particles must be commensurable. If the centroid of the particles is at the centre of the polygon, the system will be called a *central system*; if not, an *eccentric system*. We have, then, the theorem that every solution of the congruence

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-1}a^{f-1}\equiv 0,\ \text{mod. } p,$$

which is independent of p, is a central system; and conversely.

20. To prove this, take the centre of the polygon as origin, and the radius through λ_0 for the axis of x. Also take the length of this radius to be unity. Let the polar coordinates of the centroid be r, ϕ. Then

$$re^{\phi i}\,\Sigma\lambda=\lambda_0+\lambda_1 e^{\theta i}+\lambda_2 e^{2\theta i}+\ldots+\lambda_{f-1}^{(f-1)\theta i},$$

where $\theta=2\pi/f$.

Now $e^{\theta i}$ is a primitive f^{th} root of unity, so that

$$F(e^{\theta i}) = 0.$$

Hence it follows that, if $\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots$ is divisible by Fa (that is to say, if $\lambda_0, \lambda_1, \lambda_2 \ldots$ satisfies the congruence independently of p), then

$$r = 0,$$

and the system is a central system. Conversely, if the system is a central system,

$$\lambda_0+\lambda_1 e^{\theta i}+\lambda_2 e^{2\theta i}+\ldots = 0,$$

and the expression on the left must be a multiple of $F(e^{\theta i})$; because $F(e^{\theta i})$ is irreducible.

Central Systems: Particular Cases. (Arts. 21–23.)

21. When f is a prime number,

$$G\lambda = 1-\lambda,$$

so that for a central system we have

$$\lambda_0 = \lambda_1 = \lambda_2 = \ldots = \lambda_{f-1},$$

that is to say, all the particles must be of equal weight.

22. When f is the product of two different primes, say

$$f = a \,.\, b.$$

Supposing a to be the smaller factor, starting at any particle, form a clusters, each made up of a consecutive particles, the clusters being arranged symmetrically around the f-gon. All these clusters must have the same weight if the system is central. Starting at another point, we get another set of clusters which must be of equal weight, but not necessarily of the same weight as the clusters of the first set. For instance, in a hexagon the particles at the ends of any side must together weigh as much as the two particles at the ends of the opposite side.

All this is the interpretation of the equation

$$G\lambda = (1+\lambda+\lambda^2+\ldots+\lambda^{a-1})(1-\lambda^b).$$

23. When f contains powers of primes, we may write

$$f = a,\, b \ldots c \,.\, \pi \;(=f' \,.\, \pi \text{ say}),$$

where π is a product of powers of $a, b, \ldots c$ (or some of them), but

does not contain any other prime. Now, in this case, $F\lambda$, $G\lambda$ contain λ only in the powers λ^{π}, $\lambda^{2\pi}$, &c. That is to say, the relations for a central system involve

$$\lambda_h,\ \lambda_{h+\pi},\ \lambda_{h+2\pi},\ \ldots,$$

which are particles on an f'-gon. Hence it appears that, if a system on the f-gon is a central system, the particles on each f'-gon must form a central system. For example, if a dodecagon bears a central system, the particles on each of the regular hexagons in the figure must also form a central system.

This remark enables us to confine the discussion, where convenient, to the cases in which f has no square factor, without loss of generality.

Central System: General Case. (Arts. 24–33.)

24. We proceed to prove two properties of central systems when f is unrestricted in value. In expressing the first theorem it is convenient to speak of a central system on a regular polygon of a prime number of sides as a prime central system. The theorem may then be stated thus:—Every central system on an f-gon is identical with the sum of prime central systems belonging to the figure, or with the difference of two such sums.

Let the prime factors of f be $a, b, \ldots c$. Suppose the f-gon to be loaded with particles of weights $A_0, A_1, A_2, \ldots A_{f-1}$ so that the particles on each regular a-gon form a central system. This implies that the particles on each a-gon are of equal weight, so that

$$A_r = A_s \text{ if } r \equiv s, \text{ mod. } f/a.$$

Let $B, \ldots C$ have similar meanings with respect to the $b, \ldots, c$-gons. It is to be proved that, when

$$\lambda_0, \lambda_1, \ldots \lambda_{f-1}$$

form a central system, values of $A, B, \ldots C$ can be found to satisfy all the equations

$$\lambda_r = A_r + B_r + \ldots + C_r \ (r = 0, 1, 2, \ldots f-1),$$

where
$$A_r = A_s \text{ if } r \equiv s, \text{ mod. } f/a,$$
$$B_r = B_s \text{ if } r \equiv s, \text{ mod. } f/b,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$C_r = C_s \text{ if } r \equiv s, \text{ mod. } f/c.$$

Consider any linear function of $\lambda_0, \lambda_1, \lambda_2, \ldots$ which, when expressed in terms of the $A, B, \ldots C$, is free from A. Then for every λ_r in the

function there must be another λ, say λ_s, with the opposite sign, and having $s \equiv r$, mod. a' (a' written for f/a). If, then, in this linear function of the λ we change subscripts into indices, the result will be divisible by $1-\lambda^{a'}$, because it is made up of pairs of terms each of which is so divisible. Similarly, any linear combination of λ_0, λ_1, &c. which is free from $A, B, \dots C$ must, when subscripts are changed to indices, be divisible by $1-\lambda^{a'}$, $1-\lambda^{b'}$, ... $1-\lambda^{c'}$, and thus it is divisible by the lowest common multiple of $1-\lambda^{a'}$, $1-\lambda^{b'}$, ... $1-\lambda^{c'}$, which is the function $G\lambda$ of Art. 18. The order of $G\lambda$ being γ, $=f-\tau f$, it follows that the γ equations

$$\lambda_r = A_r + B_r + \dots + C_r \; (r = 0, 1, 2, \dots \gamma-1)$$

imply no relation between the λ.

When $A, B, \dots C$ have been determined so as to satisfy these γ equations, λ_γ is determined by the condition (necessary to ensure that the system $\lambda_0, \lambda_1, \dots$ is central)

$$G\lambda = 0.$$

Now $$\lambda_\gamma = A_\gamma + B_\gamma + \dots + C_\gamma$$

satisfies this equation: for it is clear that an aggregate of central systems $A, B, \dots C$ must be central. And, since $G\lambda$ is linear in λ_γ, no other value of λ_γ is possible. Similarly, it follows that

$$\lambda_r = A_r + B_r + \dots + C_r \; (r = \gamma+1, \gamma+2, \dots f-1).$$

25. The determination of $A, B, \dots C$ may be effected in the following manner:—To determine the A, eliminate the $B, \dots C$ from the γ equations of the last article. The result of the elimination is written down by forming the lowest common multiple of $\lambda^{b'}-1, \dots \lambda^{c'}-1$, multiplying this by $\lambda^0, \lambda, \lambda^2, \dots$ in turn, until the power $\lambda^{\gamma-1}$ appears. In this expression change indices into subscripts, and equate the result to the corresponding function of the A. The system of equations thus formed will be satisfied if we assign arbitrary integral values to any $f/a - \tau(f/a)$ of the A, and determine the rest suitably. To determine the B, eliminate all the $A, B, \dots C$ except A, B. The A being known, it will be found that the equations for B are satisfied when arbitrary values are assigned to $f/b - \tau(f/b) - \tau(f/ab)$ of the B (but these may not be arbitrarily selected). Similarly all the rest of the weights $A, B, \dots C$ may be determined.

26. In spite of the number of arbitrary elements in the values of $A, B, \dots C$, it is not generally possible to make them all positive. An example will suffice to prove this. Take the pentagons in a 30-gon.

One of the equations for the A $(a = 5)$ is

$$\lambda_{20}+\lambda_{15}-\lambda_5-\lambda_9 = A_2+A_3-A_5-A_0$$

(for $A_{20} = A_2$ and $A_{15} = A_3$). Now, if all the A, B, C are positive, we must have

$$A_2+A_3 \leqq \lambda_2+\lambda_3$$

(for $\lambda_2 = A_2+B_2+C_2$, $\lambda_3 = A_3+B_3+C_3$),

and

$$-A_5-A_0 \leqq 0.$$

Hence, if all the A, B, C are positive, we must have

$$\lambda_{20}+\lambda_{15}-\lambda_5-\lambda_9 \leqq \lambda_2+\lambda_3;$$

and, when this inequality does not hold, some of the A, B, C must be negative.

It is this that makes necessary the alternative statement in the theorem enunciated, Art. 24.

27. The argument of Art. 24 proves that, when particles of any weights $(\lambda_0, \lambda_1, \lambda_2, \ldots)$ have been placed at $f-\phi f$ consecutive vertices of an f-gon, there is one, and only one, central system which includes these particles. It should, however, be noted that this assumes that negative weights are admissible. The theorem cannot be extended to an arbitrarily selected set of $f-\phi f$ points (for instance, we cannot assign all the weights which actually appear in (7λ)); but there are some sets of $f-\phi f$ non-consecutive points which may be arbitrarily weighted, as we shall now show.

28. We assume that f contains no square factor, an assumption which does not really affect the generality of the result (Art. 23). Upon the f-gon select any vertex as the zero, and number the other vertices in order. A vertex whose number is a totitive of f (i.e., prime to f), being called a totitive point, we have the theorem:—When arbitrary weights are placed at the non-totitive points, the totitive points can always be weighted, and that in one way only, so that the whole may be a central system. Negative weights will generally occur, but it will be shown how the difficulties thus arising may be set aside.

29. The proof of the theorem consists in writing down and verifying an expression for the weight, λ_k, at any totitive point k, in terms of the weights at the non-totitive points.

Let A be any divisor of f; then, since f contains no square factor, A

and f/A are prime to each other, and there is therefore one, and only one, positive integer, α, less than f which satisfies the congruences

$$\alpha \equiv 0, \text{ mod. } A, \equiv t, \text{ mod. } f/A.$$

Let α be determined for every divisor, A, of f, including 1 and f. Take the results as subscripts to λ and prefix to the λ the sign $+$ or $-$ according as the divisor A contains an even or an odd number of prime factors. It will now be proved that the aggregate of λ thus formed contains only one λ, viz. λ_t, with a totitive subscript; and that it vanishes when the λ form a central system. These two properties being proved, the theorem is established; for we have λ_t expressed in terms of the weights at the non-totitive points.

The first property comes at once from observing that α is a multiple of A, and therefore cannot be a totitive of f except when $A = 1$. In this excepted case $\alpha = t$.

The divisors of f may be arranged in two classes with respect to any prime factor, a, of f. One class contains the divisors A, B, C, ... which are not multiples of a; the other consists of Aa, Ba, Ca, ... Together these make up all the divisors of f, which by hypothesis does not contain the factor a^2. To determine the subscripts α, α' belonging to the pair of divisors A, Aa, we have

$$\alpha \equiv 0, \text{ mod. } A, \equiv t, \text{ mod. } Ba,$$

$$\alpha' \equiv 0, \text{ mod. } Aa, \equiv t, \text{ mod. } B,$$

where $B = f/Aa$.

By subtraction, $\quad \alpha - \alpha' \equiv 0, \text{ mod. } A, \equiv 0, \text{ mod. } B,$

so that $\quad \alpha - \alpha' \equiv 0, \text{ mod. } AB,$

that is to say, $\quad \alpha - \alpha' \equiv 0, \text{ mod. } f/a.$

On the other hand, since

$$\alpha \equiv t, \text{ mod. } Ba, \text{ and } \alpha' \equiv 0, \text{ mod. } Aa,$$

it follows that $\quad \alpha - \alpha' \equiv t, \text{ mod. } a;$

so that α, α' cannot be equal, since t is not a multiple of a.

This pair α, α' gives to the aggregate of the λ a pair of terms

$$\lambda_\alpha - \lambda_{\alpha'}$$

with opposite signs, because Aa contains one prime factor more than

A; and it has been shown that

$$a-a' \equiv 0, \text{ mod. } f/a.$$

It follows at once that the aggregate of the λ, when indices are written instead of subscripts, is divisible by $1-\lambda^{f/a}$. Similarly, treating the other prime factors of f, it appears that the aggregate is divisible by $G\lambda$ (as in Art. 24), and therefore vanishes for a central system.

30. An attempt to form a relation between the λ for non-totitive points leads to mere identities. Suppose, for instance, t, instead of being a totitive, were a multiple of a. Then the a, a' determined as above would be equal, and the λ-aggregate would consist of vanishing pairs $\lambda_a - \lambda_a$.

31. It is only necessary to find half of the a; for, if a correspond to the divisor A, then $f+t-a$ corresponds to the divisor f/A. Moreover, when one totitive λ, preferably λ_1, has been expressed in terms of the non-totitive λ, the values of the others can be found at once by multiplying the subscripts into the several totitives in turn. If a non-totitive multiplier be used, a mere identity results, as it should.

32. An example will make this clear. Let $f = 30$. Then for λ_1 we have

$A = 1,$	2,	3,	5,	6,	10,	15,	30
$a = 1,$	16,	21,	25,	6,	10,	15,	0.

The table is divided by vertical lines into compartments. In the first of these, A contains 0 factor; in the second it contains 1 factor; in the third, 2; and in the last, 3. Thus the λ-aggregate is

$$\lambda_1-\lambda_{16}-\lambda_{21}-\lambda_{25}+\lambda_6+\lambda_{10}+\lambda_{15}-\lambda_0 = 0.$$

If we multiply the subscripts by any totitive of 30, 7 for example, we get

$$\lambda_7-\lambda_{22}-\lambda_{27}-\lambda_{25}+\lambda_{12}+\lambda_{10}+\lambda_{15}-\lambda_0 = 0,$$

which determines λ_7. But if we multiply by any non-totitives, such as 5, 6, we get the identities

$$\lambda_5-\lambda_{20}-\lambda_{15}-\lambda_5+\lambda_0+\lambda_{20}+\lambda_{15}-\lambda_0 = 0,$$

$$\lambda_6-\lambda_6-\lambda_6-\lambda_0+\lambda_6+\lambda_0+\lambda_0-\lambda_0 = 0.$$

33. It is an obvious corollary to the theorems of Arts. 24, 28, that if a polygon has $f-\tau f$ independent vertices (consecutive vertices, or

non-totitive vertices, for instance), but not all its vertices, unloaded, the system cannot be a central system.

Eccentric Systems. (Arts. 34–42.)

34. Every eccentric system is necessarily one of the non-recurring systems considered in Art. 15. For the centroids of any eccentric system, and those derived from it by cyclic substitutions, have f different positions; viz., one is in each of the sectors of the polygon. And it is plain that two systems cannot be identical if they have different centroids. It is not to be understood, and it is not a fact, that eccentric systems are the only non-recurring systems. For example: $f = 6$; 011102 is a central system which is non-recurring.

We proceed to consider the properties of systems which have their centroids at the same point, and those which have centroids on the same radius of the polygon. It is convenient to take separately the cases in which f is, and those in which f is not, a prime.

Concentric Systems: f prime.

35. Let $(\mu_0, \mu_1, \ldots)$ and $(\nu_0, \nu_1 \ldots)$ be two systems on the same f-gon which have a common centroid whose coordinates are r, ϕ. Then

$$r e^{\phi i} \Sigma\mu = \Sigma\mu_k e^{k\theta i}, \quad (\theta = 2\pi/f),$$

$$r e^{\phi i} \Sigma\nu = \Sigma\nu_k e^{k\theta i}.$$

Eliminating r,

$$\Sigma\,(\mu_k \Sigma\nu - \nu_k \Sigma\mu)\, e^{k\theta i} = 0$$

Therefore particles of weight $\mu_k \Sigma\nu - \nu_k \Sigma\nu$ form a central system, so that

$$\mu_0 \Sigma\nu - \nu_0 \Sigma\mu = \ldots = \mu_k \Sigma\nu - \nu_k \Sigma\mu = \ldots = \mu_{f-1} \Sigma\nu - \nu_{f-1} \Sigma\mu$$

$$= \frac{1}{f}\,(\Sigma\mu\,.\,\Sigma\nu - \Sigma\nu\,.\,\Sigma\mu) = 0,$$

whence
$$\mu_0 : \nu_0 = \ldots = \mu_k : \nu_k = \ldots;$$

or, what is equivalent, $(\mu_0, \mu_1, \ldots)$ and $(\nu_0, \nu_1, \ldots)$ are both multiples of a third system $(\lambda_0, \lambda_1 \ldots)$ which has its centroid at the same place; and it may be taken that

$$\lambda_0, \lambda_1, \ldots$$

have no common factor but unity. It is obvious that all multiples of $(\lambda_0, \lambda_1, \ldots)$ will be concentric with it; and it has just been proved that there are no other systems concentric with it.

36. It is to be noted that, if $(\mu_0, \mu_1, \ldots)$ satisfy the congruence of Art. 12, viz.,

$$\mu_0+\mu_1 a+\mu_2 a^2+\ldots \equiv 0, \text{ mod. } p,$$

the same is true of $(\lambda_0, \lambda_1, \ldots)$, save only when all the μ are multiples of p, a case which will be discussed hereafter (Art. 60).

Co-radial centroids. (Arts. 37–39.)

37. Next consider two systems whose centroids, G_1, G_2, are in a common radius OG_1G_2; and suppose $OG_2 > OG_1$. If OG_1, OG_2 are commensurable, the system G_1 can be derived from the system G_2 by combining a proper multiple of the set at G_2 with a central system. It follows from Art. 35 a system formed in this way so as to have its centroid at G_1 must be identical with the given system, or they must both be multiples of a system centred at G_1. It is more useful, however, to proceed outwards from G_2; by subtracting central systems from the given system G_2. This may be continued until one of the reduced weights vanishes. The centroid of the system thus obtained is an extreme point for that particular radius; and all the systems co-radial with it may be derived by adding central systems to it. It is noticeable that between O and the centroid G of the extreme system there are an infinite number of centroids, viz., a centroid at every point G_1, such that OG_1 is commensurable with OG; but beyond G there is not any centroid whose distance from O is commensurable with OG. The extreme system has an advantage in that it is instantly distinguished from a central system in which all the weights must vanish when one of them vanishes.

38. Whether there can be upon one radius two centroids G_1, G_2, such that OG_1, OG_2 are incommensurable, I cannot say. But, if so, the two sets of centroids would be as distinct as if they were in different radii.

39. When f is composite, a centroid does not belong exclusively to one system and its multiples. For, if $(\mu_0, \mu_1, \ldots)$ and $(\nu_0, \nu_1, \ldots)$ be two central systems such that $\Sigma\mu = \Sigma\nu$; and $(\lambda_0, \lambda_1 \ldots)$ be any eccentric system, then $(\lambda_0-\mu_0+\nu_0, \lambda_1-\mu_1+\nu_1, \ldots)$ is concentric with $(\lambda_0, \lambda_1, \ldots)$, and has the same total weight; but clearly it is not identical with it. And the same sort of thing happens with co-radial systems. But, in

reference to the latter, there is a remark of some interest relating to extreme systems.

Extreme systems with all weights positive. (Arts. 40–42.)

40. Let there be an eccentric system E, upon an f-gon, the vertices of which are numbered 0, 1, 2, ... in order. Let C be a central system the particles of which at the non-totitive points are of the same weight as the corresponding particles of E. Then the system $E-C$ will have its $f-rf$ non-totitive vertices unloaded, and its centroid will lie on the same radius as that of E. In general, however, at some of the totitive points $E-C$ will comprise negative weights, and so be unavailable. We shall now explain how this may be modified; and for this purpose we re-state, graphically, the results already communicated in Art. 47 of the first section of this memoir. To fix the ideas, the particular case in which $f = 15$ is discussed; but the method is quite general.

41. The totitives of 15 form a group which may be expressed, and that in one way only, as a product of two simple groups, one belonging to the factor 5, and the other to the factor 3, of 15. The decomposed form is

$$(1\,.\,7\,.\,4\,.\,13)\;(1\,.\,11).$$

The totitives may therefore be considered as distributed on two pentagons of the 15-gon, four points of each being occupied. *Cf.* Fig. 1, in which the continuous lines give the deficient pentagon,

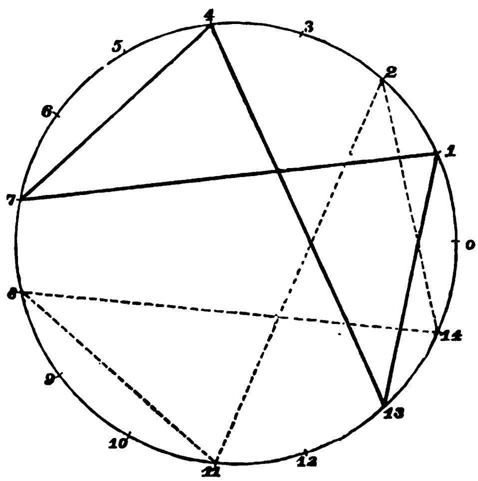


Fig. 1.

1, 7, 4, 13, and the dotted lines mark the multiple, 11, 2, 14, 8. Or, they may be considered as distributed on four equilateral triangles (two points of each being occupied). These are marked in the second figure, the continuous line indicating the group 1, 11, while the dotted lines show the multiples (7, 2), (4, 14), (13, 8).

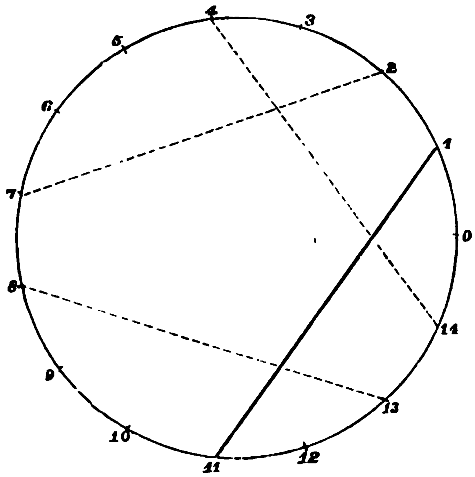


Fig. 2.

42. Suppose now that the system $E-C$ has negative weights at some of the totitive points; say it contains $-\lambda_2$, $-\lambda_4$, $-\lambda_8$, and let $\lambda_2 > \lambda_8$. If now λ_4 be added to each vertex of the pentagon 1, 4, 7, 10, 13, and λ_2 to each vertex of the pentagon 2, 5, 8, 11, 14 (these additions being two central systems), an eccentric system is obtained whose centroid is co-radial with that of E, and which is positively weighted at the points 1, 5, 7, 8, 10, 11, 13, 14.

Or, we may place a particle of weight λ_2 at each vertex of the triangle, 2, 7, 12; λ_4 at each vertex 4, 9, 14; and λ_8 at each vertex 3, 8, 13. We then get an eccentric system co-radial with E, positively weighted at the eight points 1, 3, 7, 9, 11, 12, 13, 14, and comprising no other particles.

These systems with $f-rf$ unloaded points, and the others positively weighted, are analogous to the extreme systems described in Art. 37; but the relations between the different forms require further examination.

General method of solving the fundamental congruence. (Art. 43.)

43. The method which gives numerical solutions of the congruence

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-1} a^{f-1}\equiv 0, \text{ mod. } p,$$

and the means by which it is ensured that no suitable solution is excluded, will now be explained. For the calculation of the cyclotomic, we require only those solutions

$$\lambda_0, \lambda_1, \lambda_2, \ldots \lambda_{f-1},$$

such that $\sigma, = \lambda_0+\lambda_1+\lambda_2+\ldots+\lambda_{f-1},$

does not exceed e; though, for a check on the work, it is desirable to extend the limit to $e+1$. Hence solutions in small positive integers are especially required. To obtain these we select the smallest value of a_f, and then express p, $2p$, $3p$, ... as numbers in the scale of a. The "digits" in any one of these numbers form a solution of the congruence. Supposing the multiplication table formed so far as to include all multiples of p which are less than a^f, all the solutions will be obtained in which the λ are less than a. If a is less than e, this condition may possibly exclude appropriate solutions; but the solutions excluded thus may easily be formed by replacing any consecutive λ, say λ_k, λ_{k+1} (including the case $k=f-1$, $k+1=0$) by λ_k+a, $\lambda_{k+1}-1$. This increases σ by $a-1$: and therefore the number of times the operation may be repeated is fixed beforehand. Of course, when a solution may be amplified in this way, every consecutive pair of λ must be modified, giving f new solutions.

Abbreviated methods. (Arts. 44–46.)

44. The process described enables us to determine the solutions to any proposed extent, and with no risk of omission. But, unless f and a are small, the work is very great, and only a small percentage of the solutions obtained are suitable (*i.e.*, are such that $\sigma < e+1$). As it generally happens, so in this case, many artifices serve to reduce the labour to a manageable amount. Some of these will now be noticed.

It is convenient to calculate the central systems separately. When f is a prime, these are merely 111 ..., 222 ..., 333 ..., &c. When f is a power of a prime, then the central systems are combinations of the central systems for the prime, sandwiched together. For instance, $f=9$; the central systems for $f=3$ being 000; 111; 222, &c., the central systems for $f=9$ are such as 012012012 or 021021021. When f is a product of different primes, the central

systems may be formed by using the theorem of Art. 24. Remembering that some of the A, B, C may be negative, it is necessary to examine the solutions formed by subtraction as well as by addition; and probably it will be found best to make up a special rule for each case. In the case $f = 6$, the only case I have worked with, it appears that all central systems can be formed by addition merely—and a rule is at once suggested by which the central systems can be written down in regular succession by an almost mechanical process.

45. The central systems being found as far as necessary, the multiplication table can at once be safely reduced to a fraction of its original extent. For instance, when f is a prime the table need not extend beyond α^{f-1}, since all eccentric systems can be derived by adding central systems to eccentric systems which contain at least one 0. When f is not a prime, the reduction is considerably more important.

46. One other contrivance may be mentioned. Take any two solutions whose total weights are σ_1, σ_2, respectively. If these, or any cyclic permutations of them, be added together, the result is a new solution whose sum is $\sigma_1+\sigma_2$; but if any "carrying" has been done in the addition, then for each "carrying" the new σ is reduced by $a-1$. As an example of this, the case of $p = 71$, $f = 7$, $a = 20$ may be quoted. A multiple of p is found to be 0000.1.19.1 (which means $a^2+19a+1$). Adding this to 000.1.19.1.0, the solution 0002101 is obtained. Another solution 0001.0.1.3 occurs early in the table: and from these and the central system all the suitable solutions may be compounded.

Of course, in any such short cut, there is considerable risk of omitting solutions; but such omissions are detected by trying whether the value of P_{e+1}, a coefficient of the cyclotomic, vanishes as it will do if the work has been complete.

Minimum of σ for eccentric systems. (Arts. 47, 48.)

47. When f is a prime, it sometimes happens—for instance, when $p = 31$, $f = 5$, $a = 2$—that the value of p, when expressed in the scale of a, is 111 ... 1. If in such a number two consecutive 1's are replaced by 0, $a+1$, the resulting system is an eccentric system. Hence there is an eccentric system for which

$$\sigma = a+f-1.$$

Probably this is the minimum value of σ. But, however that may be, it is easy to show that, in this case, the minimum value of σ is

greater than a. It is clear that a system for which σ is a minimum must contain at least one zero. For a system which contains no zero can be made by adding a central system to the extreme system co-radial with it. This implies that in the addition there is no "carrying": for carrying alters the radius on which the centroid lies. Suppose the system cyclically transformed so that λ_{f-1} is a zero. Then

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-2}a^{f-2}$$

is a multiple of p. But, that this may be possible, at least one of the λ must be greater than a; for, if all of them be made equal to a, the sum

$$= a+a^2+\ldots+a^{f-1}=p-1$$

by hypothesis, and therefore cannot be a multiple of p. Since, then, one of the λ is greater than a, it follows *a fortiori* that

$$\sigma > a.$$

Since $$a^f > p,$$

it follows further that $$\sigma > \sqrt[f]{p}.$$

48. There is a presumption in favour of the theorem that, when f is prime, $1+a+a^2+\ldots+a^{f-1}$ is a prime number for an infinite number of values of a; and that for composite values of f, Fa contains an infinite number of primes. If this were so, it would follow that, by taking p sufficiently large, the minimum value of σ may be made as large as we please. But the examination of a considerable number of examples seems to indicate that the inequality, $\sigma > \sqrt[f]{p}$, holds good for all values of p, and not merely for those which have the special property assumed in the last article. If this guess is right, the inequality ought to be capable of proof without making the assumption; but I have not succeeded in finding such a proof.

Determination of $\mathfrak{P}$. (Art. 49.)

49. The value of log $\mathfrak{P}$ having been determined, the value of $\mathfrak{P}$ can be found by the rules given in text-books on the theory of equations. It is unnecessary to dwell upon this further than to remark that the use of the Q form introduced in Art. 50 appears to give the reduction with less trouble than the ordinary process.

Conditions that a function given by its logarithm should have all its coefficients integral. (Arts. 50–53.)

50. For the purposes of the present paper it is essential to determine the conditions that a function, given by its logarithm,

should have all its coefficients integral, the first coefficient being 1. The function will be represented by

$$1+P_1y+P_2y^2+\dots\dots\dots\dots$$

Now let

$$1+P_1y+P_2y^2+\dots=(1-Q_1y)(1-Q_2y^2)(1-Q_3y^3)\ \dots,$$

where the $=$ is meant to express that the values of P_1, P_2, ... and of Q_1, Q_2 ... up to any assigned extent, are such that the two expressions are identical up to that extent: so that there is no question of convergence. In other words, the equation is an abbreviation of the several statements,

$$\begin{aligned} P_1 &= -Q_1 \\ P_2 &= -Q_2 \\ P_3 &= -Q_3+Q_1Q_2 \qquad\qquad (P,\ Q), \\ &\dots\dots\dots\dots\dots\dots \\ P_6 &= -Q_6+Q_1Q_5+Q_2Q_4-Q_1Q_2Q_3, \\ &\quad \&c. \end{aligned}$$

It follows that

$$\begin{aligned} &\log(1+P_1y+P_2y^2+\dots) \\ =\ &\log(1-Q_1y) = -Q_1y-Q_1^2\frac{y^2}{2}-Q_1^3\frac{y^3}{3}-Q_1^4\frac{y^4}{4}-Q_1^5\frac{y^5}{5}-Q_1^6\frac{y^6}{6}-\&c. \\ +\ &\log(1-Q_2y^2) \qquad -Q_2y^2 \qquad -Q_2^2\frac{y^4}{2} \qquad -Q_2^3\frac{y^6}{3}\dots \\ +\ &\log(1-Q_3y^3) \qquad -Q_3y^3 \qquad -Q_3^2\frac{y^6}{2}\dots \\ +\ &\log(1-Q_4y^4) \qquad -Q_4y^4 \\ +\ &\log(1-Q_5y^5) \qquad -Q_5y^5 \qquad \dots \\ +\ &\log(1-Q_6y^6) \qquad -Q_6y^6\dots \\ +\ &\&c. \qquad -\&c. \end{aligned}$$

On the other hand we have

$$\log(1+P_1y+P_2y^2+\dots) = -s_1y-s_2\frac{y^2}{2}-s_3\frac{y^3}{3}-\&c.$$

Hence

$$\begin{aligned} s_1 &= Q_1, \\ s_2 &= 2Q_2+Q_1^2, \\ s_3 &= 3Q_3+Q_1, \end{aligned}$$

$$s_4 = 4Q_4 + 2Q_2^2 + Q_1^4, \qquad (s, Q),$$

$$s_5 = 5Q_5 + Q_1^5,$$

$$s_6 = 6Q_6 + 3Q_3^2 + 2Q_2^3 + Q_1^6,$$

and, generally, $$s_n = nQ_n + \ldots + \delta Q_\delta^{\delta'} + \ldots + Q_1^n,$$

where δ, δ' are integers, such that $\delta\delta' = n$, and the expression includes all values of δ.

51. From the equations (P, Q) it is seen that, if $Q_1, Q_2 \ldots Q_n$ are integers, then $P_1, P_2 \ldots P_n$ are likewise integers. From the equations (s, Q) it appears that, for $Q_1 \ldots Q_n$ to be integers, it is necessary (but not sufficient) that $s_1 \ldots s_n$ should be integers. Suppose that $Q_1 \ldots Q_{n-1}$ are all integral; then, in order that Q_n may be integral, it is necessary that s_n when divided by n should leave a certain remainder which is determined by the values of some of the Q of lower rank. This remainder is, however, more simply expressed in terms of the s of lower rank; and it comes out that the relations necessary and sufficient to ensure that the Q are integers are such as

$$s_{ab} - s_a - s_b + s_1 \equiv 0, \text{ mod. } ab,$$

$$s_{a^2b} - s_{ab} - s_{a^2} + s_a \equiv 0, \text{ mod. } a^2b,$$

where a, b are primes. The rule for forming the critical expression for s_n is this. Divide n by each of its prime factors in turn; use the quotients as subscripts to s; and denote the aggregate of the s thus found by Σ_1. Again, divide n by the product of every pair of different prime factors. The aggregate of the s whose subscripts are the quotients may be called Σ_2. The process is to be continued as far as possible. Then the critical expression is

$$s_n - \Sigma_1 + \Sigma_2 - \Sigma_3 + \ldots,$$

and, if this is divisible by n, Q_n is integral (presuming that the preceding Q are all integral).

The analogy of this rule to that for forming the function Fa is too striking to escape notice.

52. The proof comes from considering the value of the critical expression when written in terms of Q. For instance,

$$s_{ab} - s_a - s_b + s_1,$$

in terms of Q, is

$$ab\,.\,Q_{ab} + a\,(Q_a^b - Q_a) + b\,(Q_b^a - Q_b) + Q_1^{ab} - Q_1^b - Q_1^a + Q_1.$$

Each of the first three terms is seen to be divisible by ab. The part containing Q_1 may be written

$$(Q_1^b)^a - Q_1^b - \{Q_1^a - Q_1\},$$

$$(Q_1^a)^b - Q_1^a - \{Q_1^b - Q_1\},$$

so that it is divisible by a, and by b, and therefore by ab. If, then, Q_{ab}, Q_a, Q_b, Q_1 are integral, $s_{ab} - s_a - s_b + s_1$ must be divisible by ab; and, conversely, if Q_a, Q_b, Q_1 are integral and $s_{ab} - s_a - s_b + s_1$ divisible by ab, Q_{ab} is integral.

Again, consider the example

$$s_{a^2b} - s_{ab} - s_{a^2} + s_a,$$

which is

$$= a^2bQ_{a^2b} + a^2(Q_{a^2}^b - Q_{a^2}) + ab(Q_{ab}^a - Q_{ab})$$
$$+ a(Q_a^{ab} - Q_a^b - Q_a^a + Q_a)$$
$$+ Q_1^{a^2b} - Q_1^{ab} - Q_1^{a^2} + Q_1^a.$$

The parts involving Q_{a^2b}, Q_{ab}, Q_{a^2}, Q_a are seen to be divisible by a^2b, as in the first example. The part containing Q_1 is divisible by b, for it may be written

$$(Q_1^{a^2})^b - Q_1^{a^2} - \{(Q_1^a)^b - Q_1^a\},$$

and it is also divisible by a^2, for

$$Q^{a^2} \equiv Q^a, \text{ mod. } a^2, \text{ and } Q^{a^2b} \equiv Q^{ab} \text{ mod. } a^2,$$

by reason of the generalized Fermat's theorem.

53. These examples suggest, what is obvious when the suggestion has been made, that, when a is prime, the part involving Q_{ah} in the expression for ahk is (omitting the external multipliers) of the same form as the part involving Q_h in the hk function, and by repeating the reduction it is of the same form as the part involving Q_1 in the critical expression for k. Hence it is only needful to consider the part of the critical expression which involves Q_1; and this part is written down by merely changing the subscripts of the critical expression into indices. When this is done the divisibility by n is seen at once by using the generalization of Fermat's theorem. For let $n = a^a A$, where a is prime and A is prime to a. Suppose one of the divisors of n formed by our rule to be $a^e B$, so that A/B contains no square factor. Then $a^{e-1}B$ is another divisor of n, also furnished by the rule, and in the critical expression we have a pair of terms,

$$\pm (Q_1^{a^e B} - Q_1^{a^{e-1} B}).$$

This is divisible by a^α. As the whole expression can be expressed in pairs of this nature, it also is divisible by every prime-power in n; and therefore by n.

Application to series comprised in $\log \mathfrak{P}$. (Arts. 54–57.)

54. I proceed to show that, if $\lambda, \mu, \dots \nu$ (written for the $\lambda_0, \lambda_1, \dots$ previously used) be any positive integers whose greatest common measure is unity and whose sum is σ, then

$$s_n = \frac{n\sigma!}{n\lambda!\, n\mu! \dots n\nu!} \cdot \frac{1}{\sigma}, = \frac{(n\sigma-1)!\, n}{n\lambda!\, n\mu! \dots n\nu!},$$

satisfies the conditions obtained in Art. 51.

55. It is easy to see that s_n is integral. For it may be written

$$\frac{(n\sigma-1)!}{n\lambda!\, n\mu! \dots (n\nu-1)!} \cdot \frac{n}{n\nu} = \frac{N}{\nu},$$

where N is some integer. Similarly, it is $= L/\lambda$, $M/\mu, \dots$ where $L, M, \dots$ are integers. But the greatest common measure of $\lambda, \mu, \dots \nu$ is 1, so that the denominator of these equal fractions when in lowest terms must also be 1; that is to say, s_n is an integer.

56. We shall now prove that

$$s_{a^\alpha A}(1+ha^\alpha) = s_{a^{\alpha-1}A}(1+ka^\alpha),$$

where a is prime, A prime to a, and h, k are integers.

This at once gives $\quad s_{a^\alpha A} \equiv s_{a^{\alpha-1}A}$, mod. a^α;

and hence, as in Art. 53, it follows that the critical expression is divisible by n.

It is convenient to write B for $a^{\alpha-1}A$ in some places, and the theorem to be proved may then be written

$$s_{aB}(1+ha^\alpha) = s_B(1+ka^\alpha).$$

Consider the expression $(aB\rho)!$ It contains multiples of a, the product of which is

$$a \,.\, 2a \,.\, 3a \dots B\rho a = a^{B\rho} \,.\, (B\rho)!$$

The remaining factors may be arranged in $A\rho$ products of which the first is

$$1 \,.\, 2, \dots (a-1)(a+1) \dots (a^\alpha-1),$$

the second is formed by adding a^s to each factor of the first; the third by adding $2a^s$, and so on. Now, by Gauss' generalization of Wilson's theorem, the first product

$$= -1 + ma^s,$$

where m is an integer. Clearly the other products are of the same form. Hence the product of all the factors of $(aB\rho)!$ which are prime to a

$$= (-1)^{A\rho} + ma^s,$$

and $$(aB\rho)! = a^{B\rho} . (B\rho)! \{-1)^{A\rho} + ma^s\},$$

where m is some integer.

Hence, if ρ be replaced by $\sigma, \lambda, \mu, \ldots \nu$, in turn, we shall have

$$s_{aB} = \frac{1}{\sigma} \cdot \frac{(aB\sigma)!}{(aB\lambda)! \ldots (aB\nu)!} = \frac{1}{\sigma} \cdot \frac{(B\sigma)!}{(B\lambda)! \ldots (B\nu)!} \cdot \frac{(-1)^{A\sigma} + ka^s}{(-1)^{A\sigma} + ha^s},$$

for it is at once seen that powers of a cancel. But this is

$$s_{aB} . (1 + ha^s) = s_B . (1 + ka^s),$$

where h, k are positive or negative integers.

This result proves that the congruence

$$s_{aB} \equiv s_B, \text{ mod. } a^s,$$

remains true when the sides are divided by any common factor, whether prime to a or not.

To sum up, we may ennunciate the results in the theorem:

If $\lambda, \mu, \ldots \nu$ be any positive integers whose greatest common measure is 1, and whose sum is σ, and

$$s_n = \frac{1}{\sigma} \cdot \frac{\sigma n!}{\lambda n! \ldots \nu n!},$$

then $$\exp. (-\Sigma s_n y^n/n) = 1 + P_1 y + P_2 y^2 + \ldots,$$

where $P_1, P_2, \ldots$ are integers; and if $s_1, s_2, \ldots s_n, \ldots$ have any common factor, say m, then

$$\exp. (-\Sigma s_n y^n/n) = (1 + P_1 y + P_2 y^2 + \ldots)^m,$$

where $P_1, P_2, \ldots$ are still integers.

57. When the greatest common measure of $\lambda, \mu, \ldots \nu$ is p, the theorem of Art. 54 is not true; but in this case ps_n is proved to be integral as in that article. The following articles are not affected by

the change: so that we have the theorem that, when the greatest common measure of $\lambda, \mu, \dots \nu$ is p, then

$$\exp. \left(-\Sigma p\, s_n y^n / n\right) = 1 + P_1 y + P_2 y^2 + \dots,$$

where $P_1, P_2, \dots$ are integers not divisible by p.

The eccentric factor of $\mathfrak{P}$. (Arts. 58, 59.)

58. Let $$\lambda, \mu, \dots \nu$$

be any eccentric system, such that $\lambda, \mu, \dots \nu$ have 1 for their greatest common measure. This system, with its multiples $(n\lambda, n\mu, \dots n\nu)$, contributes to $\log \mathfrak{P}$ the terms

$$-\frac{p}{f} . \Sigma \frac{1}{\sigma} . \frac{n\sigma!}{n\lambda! \dots n\nu!} . \frac{(y^\sigma)^n}{n}.$$

If we combine with this the systems formed from it by cyclic substitutions, the effect is to multiply this by f, and the total contribution is

$$-p . \Sigma \frac{1}{\sigma} . \frac{n\sigma!}{n\lambda! \dots n\nu!} . \frac{(y^\sigma)^n}{n}.$$

This may be regarded as belonging to the point G, which is the common centroid of $(\lambda, \mu, \dots \nu)$ and its multiples.

In $\mathfrak{P}$ there will therefore be a "factor,"

$$(1 - E_1 y^\sigma - E_2 y^{2\sigma} - \text{\&c.})^p,$$

where the E_1, E_2, &c. are integers; and this factor may be considered to belong to the particular point G mentioned above.

59. Every point G, in one sector of the polygon, which is the centroid of a solution of the oft-quoted congruence, contributes a factor of the same kind. For our present purpose it is unnecessary to keep these distinct; and we shall write the "eccentric factor" of $\mathfrak{P}$ in the form

$$\mathfrak{E}^p = (1 - E_k y^k - E_{k+1} y^{k+1} - \dots)^p,$$

where E_k is the first E that does not vanish; so that k is, in fact, the minimum value of σ for an eccentric system.

This may also be written

$$1 - p\,(E_k y^k + E_{k+1} y^{k+1} + \dots),$$

where E_k are still integral; and, in the beginning, positive integers.

The "third factor" of $\mathfrak{P}$. (Arts. 60, 61.)

60. Next consider the case in which $\lambda, \mu, \ldots \nu$ are all multiples of p, and therefore satisfy the congruence

$$\lambda + \mu a + \ldots + \nu a^{f-1} \equiv 0, \text{ mod. } p.$$

But it is supposed that no system of integers

$$\lambda/\delta, \ \mu/\delta, \ \ldots \ \nu/\delta$$

satisfies this congruence. Then the greatest common measure of $\lambda, \mu, \ldots \nu$ must be p.

The contribution to $\log \mathfrak{P}$ made by such a system, and the systems symmetrical with it, and their multiples, is

$$-p \cdot \Sigma \frac{1}{\sigma} \cdot \frac{n\sigma!}{n\lambda! \ldots n\nu!} \cdot \frac{(y^{\sigma})^n}{n}.$$

But in reducing from the logarithmic form we find that, as remarked in Art. 57, the factor contributed to $\mathfrak{P}$ is of the form

$$1 - \Sigma . E_{\sigma} y^{\sigma},$$

where the E_{σ} are integers not divisible by p.

As in the case of the eccentric factor, we express the product of all the individual factors in one; which is the "third factor" of $\mathfrak{P}$.

61. From the way in which the third factor originates, it is clear that all the σ are multiples of p. If the system

$$(\lambda, \mu, \ldots \nu) = (\lambda' p, \mu' p, \ldots \nu' p),$$

and $$\sigma = \sigma' p,$$

it will be seen that the f numbers

$$\lambda', \mu', \ldots \nu'$$

may be any partition of any integer σ', except those which satisfy the fundamental congruence.

The third factor is of the form

$$1 - y^p - \&c.,$$

and the coefficients of the cyclotomic are not affected by the omission of this factor.

The "central factor" of $\mathfrak{P}$. (Arts. 62–66.)

62. The general form of the contribution to $\log \mathfrak{P}$ by a central

system, and its multiples, together with the cyclic transformations, is

$$-\frac{p}{g}\cdot\Sigma\frac{1}{g\sigma}\cdot\frac{(g\sigma n)}{(\lambda n!\,\ldots\,\nu n!)^{g}}\cdot\frac{(y^{g\sigma})^{n}}{n},$$

where $\sigma=\lambda+\mu+\ldots+\nu$; so that $g\sigma$ corresponds to the σ of the preceding articles, since the sum of the denominator elements is $(\lambda+\mu+\ldots+\nu)\,g$.

63. Now the expression $\dfrac{g\sigma n!}{(\lambda n!\,\ldots\,\nu n!)^{g}}$

is divisible by $(g!)^{h}$, where h is the number of quantities $\lambda,\mu,\ldots\nu$.

For we have

$$\frac{(k\lambda+1)\,(k\lambda+2)\,\ldots\,(k\lambda+\lambda-1)}{1\quad .\quad 2\quad \ldots\quad \lambda-1}\cdot\frac{(k+1)\,\lambda}{\lambda}$$

$$=\text{an integer}\times(k+1).$$

Putting herein $k=0,1,2,\ldots(g-1)$, and taking the product of the results, we find that

$$\frac{\lambda g!}{(\lambda!)^{g}}\text{ is divisible by }g!$$

Multiplying by the integer $(\sigma g)!/(\sigma-\lambda)\,g!$ it follows that

$$\frac{\sigma g!}{(\sigma g-\lambda g)!\,(\lambda!)^{g}}\text{ is divisible by }g!$$

Similarly, $$\frac{(\sigma g-\lambda g)!}{(\sigma g-\lambda g-\mu g)!\,(\mu!)^{g}}\text{ is divisible by }g!$$

Continuing the process, and multiplying the results, we have

$$\frac{\sigma g!}{(\lambda!\,\mu!\,\ldots\,\nu!)^{g}}\text{ is divisible by }(g!)^{h},$$

where $\sigma=\lambda+\mu+\ldots+\nu$, and h is the number of quantities $\lambda,\mu,\ldots\nu$.

This result is evidently not affected when for $\sigma,\lambda,\mu,\ldots\nu$ we write $n\sigma,n\lambda,\ldots n\nu$, respectively, and the theorem is proved.

64. Hence it follows that

$$\frac{1}{g\sigma}\cdot\frac{(g\sigma n)!}{(\lambda n!\,\ldots\,\nu n!)^{g}},$$

which is integral, is divisible by

$$\frac{1}{\sigma}(g!)^{h-1}(g-1)!$$

if this is an integer, or, generally, by the numerator of this fraction when expressed in lowest terms.

Represent this divisor by d. Then in the contribution to log $\mathfrak{P}$, namely,

$$-\frac{p}{g}\Sigma\frac{1}{g\sigma}\cdot\frac{(g\sigma n)!}{(\lambda n!\ \ldots\ \nu n!)^{g}}\cdot\frac{(y^{g\sigma})^n}{n},$$

the coefficients of $y^{g\sigma n}/n$ are integers or fractions, according as d is or is not a multiple of g. And, accordingly, in $\mathfrak{P}$ there will be a corresponding factor which is an integral or fractional power of a series,

$$1+C_1y^{\sigma}+C_2y^{2\sigma}+\ldots,$$

where the coefficients C_1, C_2, ... are integers. In fact, the denominator of the fractional exponent will be the denominator of the fraction d/g when reduced to its lowest terms.

65. We can now indicate some of the characters of the product of the central factors of $\mathfrak{P}$: and firstly in the case when f is prime. The central systems are all multiples of (111 ... 1), so that

$$g=f,\quad h=1,\quad \sigma=1,$$

and therefore $$d=(f-1)!$$

Thus the central part of log $\mathfrak{P}$ is

$$-\frac{p}{f}\cdot(f-1)!\,\Sigma s'_n y^{n f/n},$$

where the Σ covers the logarithm of a series, say

$$1+C_1y^f+C_2y^{2f}+\ldots,$$

with integral coefficients. The central part of $\mathfrak{P}$ is this series raised to the power

$$-\frac{p}{f}\cdot(f-1)!,\quad=-e\,.\,(f-1)!-\frac{(f-1)!+1}{f}-\frac{1}{f},$$

whereof the last term only is a fraction.

It appears at once that the central factors of $\mathfrak{P}$ are functions of e only in the form $e\,.\,(f-1)!$ In the tables which follow will be found verifications of this.

It also appears that $\mathfrak{P}$, which has all its coefficients integral, contains two factors, (and no more),

$$(1-fy)^{-1/f} \text{ and } (1+C_1y^f+C_2y^{2f}+\ldots)^{-1/f},$$

whose coefficients are not all integers. The product of these must, therefore, have integral coefficients. Therefore

$$(1-fy)^{-1/f}(1+C_1y^f+C_2y^{2f}+\ldots)^{-1/f} = (1+y+B_2y^2+\ldots),$$

where for B_1 is written its obvious value, 1. It will be found that, if we replace

$$1+C_1y^f+C_2y^{2f}+\ldots$$

by its equivalent

$$(1-fy)^{-1}(1+y+B_2y^2+\ldots)^{-f},$$

the asymptotic factor of

$$\mathfrak{P} = (1-fy)^e(1+y+B_2y^2+\ldots)^p,^*$$

and other forms may be obtained in which both the C and the B series appear. In the tables only one combination is shown, viz., the product of all the central factors into the factor $(1-fy)^{-1/f}$.

66. The significance of the result of Art. 64 is seen in dealing with the cases in which f is or contains a power (higher than first) of a prime. It will suffice to give a particular example. Take $f = 9$. Then $g = 9, 3$, $h = 1, 3$. (The value $g = 1$ is excluded because in a central system on a nonagon the weights cannot all be different; every triangle must have equal weights at its corners). The corresponding values of σ are 1 and an indeterminate number respectively. For when $g = 9$ the system must be a multiple of $(1)^9$, but when $g = 3$ the system is a multiple of $(a, b, c)^3$, where a, b, c are any positive integers. The values of d/g in the two cases are $8! \div 9$ and $3!\,3!\,2! \div 3\sigma = 24/\sigma$. The first is an integer, and the second would also be integral but for the σ: and $\mathfrak{P}$ would contain one factor only, viz., $(1-9y)^{-\frac{1}{9}}$ raised to a fractional power.

* From this a well-known theorem relative to cyclotomics is easily deduced. The proof for unrestricted values of f may be given, as it is very short. We have

$$\mathfrak{P} = (1-fy)^{-1/f}H^{pf}\,.\,T,$$

where H and T are functions such as $1+H_1y+H_2y^2+\ldots$ with integral coefficients, and T_1, $T_2\ldots$ are multiples of p. Hence, separating the fractional-power factors,

$$(1-fy)^{-1/f}\,.\,H^{1/f} = K,$$

another function of the same kind as H.

Eliminating H, $\qquad \mathfrak{P} = (1-fy)^e\,.\,K^p\,.\,T.$

Hence $\qquad \mathfrak{P} \equiv (1-fy)^e, \text{ mod. } p,$

or $\qquad \eta^e+\eta^{e-1}+P_2\eta^{e-2}+\ldots \equiv (\eta-f)^e, \text{ mod. } p, \quad (\eta = 1/y).$

Example of Calculation of the Asymptotic Factor.

67. An abstract of the calculation of the asymptotic factor of $\mathfrak{P}$ fo the case $f = 3$ follows. This will serve to point out the interestin character of the coefficients Q (Arts. 7, 50). It will also indicate th origin of the form of the successive coefficients in the asymptoti factor described in Art. 3. The proof that this form is general, come at once from the observation that s_k is linear in p (or e) when n is no prime to f; and does not contain p (or e) when n is prime to f. Th equation (s, Q) and (P, Q) of Art. 50 then give P, the coefficient in the asymptotic factor, in the form described (Art. 3).

The results of the calculation are entered, as they are obtained, i a table such as the following :—

k	s_k	Q_k	P_k
1	-1	-1	1
2	-3	-2	2
3	$-9+2p, = -7+6e$	$-2+2e$	$4-2e$
4	-27	-9	$11-2e$
5	-81	-16	$29-4e$
6	$-243+30p$ $= -213+90e$	$-35+19e-2e^2$	$73-23e+2e^2$
7	-729	-104	$207-37e+2e^2$
8	-2187	-318	$574-88e+4e^2$
9	$-6561+560p,$ $= -6001+1680e$	$\frac{1}{3}(-1992+536e$ $+24e^2-8e^3)$	$\frac{1}{3}(4626-1178e$ $+114e^2-4e^3)$

To show the mode of forming this table, take the last line. s_9 consists of two parts, viz., $-3^8, = -6561$, and a part arising from the central system (3, 3, 3) of weight 9. The latter is

$$\frac{p}{3}\cdot\frac{9!}{3!\,3!\,3!} = 560p = 560+1680e.$$

Thus $$s_9 = -6561+560p = -6001+1680e.$$

Q_9 is given by the equation

$$s_9 = 9Q_9+3Q_3^3+Q_1^9,$$

therefore

$$\begin{array}{rcl} 9Q_9 = & s_9 & = -6001+1680e \\ & -3Q_3^3 & +\ \ 24-\ \ 72e+72e^2-24e^3 \\ & -\ Q_1^9 & +\ \ 1 \\ & & = -5976+1608e+72e^2-24e^3, \end{array}$$

or
$$Q_9 = \tfrac{1}{3}(-1992+536e+24e^2-8e^3).$$

Finally,
$$\begin{array}{rl} P_9 = -Q_9 & = \tfrac{1}{3}(1992-\ 536e-\ 24e^2+8e^3) \\ +Q_1Q_8 & +\ 318 \\ +Q_2Q_7 & +\ 208 \\ +Q_3Q_6 & +\ \ 70-\ 108e+\ 42e^2-4e^3 \\ +Q_4Q_5 & +\ 144 \\ -Q_1Q_2Q_6 & +\ \ 70-\ \ 38e+\ \ 4e^2 \\ -Q_1Q_3Q_5 & +\ \ 32-\ \ 32e \\ -Q_2Q_4Q_6 & +\ \ 36-\ \ 36e \\ & = \tfrac{1}{3}(4626-1178e+114e^2-4e^3). \end{array}$$

Tables.

Appended are some tables for $f = 3, 4, 5, 6, 7, 8, 9$ illustrative of the preceding paper. They are similar in arrangement.

Under I. are given the formulæ for the coefficients of y^0, y^1, y^2, in the asymptotic cyclotomic. The coefficients are ranged in order down the several columns. They are expressed in terms of $e = (p-1)/f$.

Under II., the same coefficients are expressed in terms of ϵ, a suitable multiple of e.

Under III. are given the numerical values of the coefficients for values of p less than 100. The end of the cyclotomic is marked by a | : and the first coefficient that is affected by the eccentric factor is underlined.

Under IV. are given the eccentric factors corresponding to the same values of p. The product of III. into IV. gives the cyclotomic (as far as y^{p-1}).

In the last column of III. references are given, such as R. p. 7, to

the page in Reuschle's *Tafeln*, on which the corresponding cyclotomic is given.

$f = 3.$

I. $1, \; -(2e-4), \;\; 2e^2-23e+73, \;\; -\tfrac{1}{3}(4e^3-114e^2+1178e-4626),$

$1, \; -(2e-11), \;\; 2e^2-37e+207, \;\; -\tfrac{1}{3}(4e^3-156e^2+2297e-13305),$

$2, \; -(4e-29), \;\; 4e^2-88e+574, \;\; -\tfrac{1}{3}(8e^3-354e^2+5869e-37707).$

$$\epsilon = 2e.$$

II. $1, \; -(\epsilon-4), \;\; \tfrac{1}{2}(\epsilon^2-23\epsilon+146), \;\; -\tfrac{1}{6}(\epsilon^3-57\epsilon^2+1178\epsilon-9252),$

$1, \; -(\epsilon-11), \;\; \tfrac{1}{2}(\epsilon^2-37\epsilon+414), \;\; -\tfrac{1}{6}(\epsilon^3-78\epsilon^2+2297\epsilon-26610),$

$2, \; -(2\epsilon-29), \;\; \tfrac{1}{2}(2\epsilon^2-88\epsilon+1148), \;\; -\tfrac{1}{6}(2\epsilon^3-177\epsilon^2+5869\epsilon-75414),$

III.

<table>
<tr><td>p</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>7</td><td>1,</td><td>1,</td><td>2 |</td><td>0,</td><td>$\underline{7}$,</td><td>21,</td><td>35,</td><td>141,</td><td>414,</td><td>898,</td><td>3101,</td><td>9107,</td><td>R. p.</td></tr>
<tr><td>13</td><td>1,</td><td>1,</td><td>2,</td><td>− 4,</td><td>3 |</td><td>$\underline{13}$,</td><td>13,</td><td>91,</td><td>286,</td><td>494,</td><td>2119,</td><td>6461,</td><td>R. p.</td></tr>
<tr><td>19</td><td>1,</td><td>1,</td><td>2,</td><td>− 8,</td><td>− 1,</td><td>5,</td><td>7 |</td><td>$\underline{57}$,</td><td>190,</td><td>266,</td><td>1425,</td><td>4503,</td><td>R. p.</td></tr>
<tr><td>31</td><td>1,</td><td>1,</td><td>2,</td><td>−16,</td><td>− 9,</td><td>−11,</td><td>43,</td><td>$\underline{37}$,</td><td>94,</td><td>82,</td><td>645 |</td><td>2139,</td><td>R. p.</td></tr>
<tr><td>37</td><td>1,</td><td>1,</td><td>2,</td><td>−20,</td><td>−13,</td><td>−19,</td><td>85,</td><td>51,</td><td>94,</td><td>− 2,</td><td>$\underline{431}$,</td><td>1477,</td><td>R. p.</td></tr>
<tr><td>43</td><td>1,</td><td>1,</td><td>2,</td><td>−24,</td><td>−17,</td><td>−27,</td><td>143,</td><td>81,</td><td>$\underline{126}$,</td><td>− 166,</td><td>249,</td><td>991,</td><td>R. p.</td></tr>
<tr><td>61</td><td>1,</td><td>1,</td><td>2,</td><td>−36,</td><td>−29,</td><td>−51,</td><td>413,</td><td>267,</td><td>414,</td><td>−1778,</td><td>− 745,</td><td>− 691,</td><td>R. p.</td></tr>
<tr><td>67</td><td>1,</td><td>1,</td><td>2,</td><td>−40,</td><td>−33,</td><td>−59,</td><td>535,</td><td>361,</td><td>574,</td><td>−2902,</td><td>−1439,</td><td>−$\underline{1753}$,</td><td></td></tr>
<tr><td>73</td><td>1,</td><td>1,</td><td>2,</td><td>−44,</td><td>−37,</td><td>−67,</td><td>673,</td><td>471,</td><td>766,</td><td>−4426,</td><td>−$\underline{2421}$,</td><td>−3279,</td><td></td></tr>
<tr><td>79</td><td>1,</td><td>1,</td><td>2,</td><td>−48,</td><td>−41,</td><td>−75,</td><td>827,</td><td>597,</td><td>990,</td><td>−6414,</td><td>−3775,</td><td>−5457,</td><td></td></tr>
<tr><td>97</td><td>1,</td><td>1,</td><td>2,</td><td>−60,</td><td>−53,</td><td>−99,</td><td>1385,</td><td>1071,</td><td>1854,</td><td>−15802,</td><td>−10509,</td><td>−16983,</td><td></td></tr>
</table>

IV.

$p = 31$	$1-31\,(y^7+36y^{10}+42y^{11}+0\,.\,y^{12}+\ldots)$
37	$1-37\,(12y^{10}+30y^{11}+y^{12}+\ldots)$
43	$1-43\,(y^8+45y^{11}+132y^{13}+1430y^{14}+\ldots)$
61	$1-61\,(55y^{18}+\ldots)$
67	$1-67\,(5y^{11}+286y^{14}+\ldots)$
73	$1-73\,(y^{10}+66y^{18}+\ldots)$
79	$1-79\,(22y^{18}+\ldots)$
97	$1-97\,(26y^{14}+\ldots)$

$f = 4.$

$$1, \qquad (2e-7)(e-3), \qquad \tfrac{1}{6}(4e^4-124e^3+1511e^2-8693e+20262),$$

$$1, \qquad 2e^2-23e+77, \qquad \tfrac{1}{6}(4e^4-164e^3+2723e^2-22189e+76896).$$

$$-(2e-2), \quad -\tfrac{1}{3}(4e^3-66e^2+380e-771),$$

$$-(2e-7), \quad -\tfrac{1}{3}(4e^3-96e^2+851e-2889),$$

$$\epsilon = 2e.$$

$$1, \qquad \tfrac{1}{2}(\epsilon-6)(\epsilon-7), \qquad \frac{1}{4!}(\epsilon^4-62\epsilon^3+1511\epsilon^2-17386\epsilon+81048),$$

$$1, \qquad \tfrac{1}{2}(\epsilon^2-23\epsilon+154), \qquad \frac{1}{4!}(\epsilon^4-82\epsilon^3+2723\epsilon^2-44378\epsilon+307584).$$

$$-(\epsilon-2), \quad \tfrac{1}{6}(\epsilon^3-33\epsilon+380\epsilon-1542),$$

$$-(\epsilon-7), \quad \tfrac{1}{6}(\epsilon^3-48\epsilon+851\epsilon-5778),$$

= 5	1,	1, $\mid$	0,	$\underline{5}$,	10,	56,	151,	710,	2160,	9545,	R. p. 2
13	1,	1,	− 4,	1 $\mid$	0,	$\underline{26}$,	39,				R. p. 15
17	1,	1,	− 6,	− 1,	1 $\mid$	$\underline{17}$,					R. p. 19
29	1,	1,	−12,	− 7,	28,	14,	− 9,	$\underline{88}$ $\mid$			R. p. 35
37	1,	1,	−16,	−11,	66,	32,	− 73,	$\underline{30}$,	44,	741 $\mid$	R. p. 51
41	1,	1,	−18,	−13,	91,	47,	−143,	− 7,	72,	$\underline{551}$,	R. p. 58
53	1,	1,	−24,	−19,	190,	116,	−601,	−246,	738,	$\underline{427}$,	R. p. 79
61	1,	1,	−28,	−23,	276,	182,	−1193,	−592,	2307,	956,	R. p. 89
73	1,	1,	−34,	−29,	435,	311,	−2659,	−1539,	7838,	3867,	
89	1,	1,	−42,	−37,	707,	543,	−6079,	−3987,	28512,	16237,	
97	1,	1,	−46,	−41,	861,	687,	−8543,	−5845,	48069,	28796,	

IV.

$p = 29$	$1-29\,(3y^7+\dots)$
37	$1-37\,(y^7+36y^9+\dots)$
41	$1-41\,(14y^9+y^{10}+\dots)$
53	$1-53\,(4y^9+165y^{11}+3960y^{13}+\dots)$
61	$1-61\,(42y^{11}+\dots)$
73	$1-73\,(15y^{11}+\dots)$
89	$1-89\,(99y^{13}+\dots)$
97	$1-97\,(55y^{13}+\dots)$

$f = 5.$

$$\begin{array}{lll}
\text{I.}\quad 1, & -(24e-180), & 288e^2-15660e+331282, \\
1, & -(24e-796), & 288e^2-30444e+1544418, \\
3, & -(72e-3532), & 864e^2-118788e+7211960, \\
11, & -(264e-15906), & 3168e^2-506484e+33850952, \\
44, & -(1056e-72490), & 12672e^2-2238720e+159612948.
\end{array}$$

$$\epsilon = 4!\, e.$$

$$\begin{array}{lll}
\text{II.}\quad 1, & -(\epsilon-180), & \tfrac{1}{2}(\epsilon^2-1305\epsilon+662564), \\
1, & -(\epsilon-796), & \tfrac{1}{2}(\epsilon^2-2537\epsilon+3088836), \\
3, & -(3\epsilon-3532), & \tfrac{1}{2}(3\epsilon^2-9899\epsilon+14423920), \\
11, & -(11\epsilon-15906), & \tfrac{1}{2}(11\epsilon^2-42207\epsilon+67701904), \\
44, & -(44\epsilon-72490), & \tfrac{1}{2}(44\epsilon^2-186560\epsilon+319225896).
\end{array}$$

III.

$$\begin{array}{r|l}
= 11 & 1, 1, 3 \mid \underline{11}, 44, \ 132, 748, \ 3388, 15378, 70378, 301114, \quad \text{R.} \\
31 & 1, 1, 3, \ 11, 44, \ 36, \underline{652} \mid 3100, \quad \text{R.} \\
41 & 1, 1, 3, \ 11, 44, \ -\underline{12}, 604, \ 2956, 13794 \mid 41\times 1562, \quad \text{R.} \\
61 & 1, 1, 3, \ 11, \underline{44}, \ -108, 508, \ 2668, 12738, 59818, 184834, \\
 & \qquad 1220562, 5910920, \quad \text{R.} \\
71 & 1, 1, 3, \ 11, 44, \ -156, \underline{460}, \ 2524, 12210, 57706, 168490, \\
 & \qquad 1174650, 5718272, 27381104, 130754580. \quad \text{R.}
\end{array}$$

$$\begin{array}{r|l}
\text{IV.}\quad p = 31 & 1-31\,(20y^6+80y^7+\dots), \\
41 & 1-41\,(y^5+10y^6+63y^7+231y^8+988y^9+\dots), \\
61 & 1-61\,(y^4+30y^7+148y^8+848y^9+882y^{10}+10980y^{11}+57917y^{12}+\dots) \\
71 & 1-71\,(10y^6+15y^7+105y^8+616y^9+792y^{10}+13660y^{11} \\
 & \qquad +50790y^{12}+242328y^{13}+1149635y^{14}+\dots
\end{array}$$

$f = 6.$

$$\begin{array}{lll}
\text{I.}\quad 1, & \tfrac{1}{2}(e-3)(9e-44), & \tfrac{1}{8}(27e^4-1254e^3+20469e^2-146818e+39 \\
1, & \tfrac{1}{2}(33e^2-297e+670), & \\
-(3e-3), & -\tfrac{1}{2}(9e^3-202e^2+1467e-3472), & \\
-(7e-14), & -\tfrac{1}{2}(45e^3-961e^2+7160e-18454). &
\end{array}$$

p		
$p = 7$	1, 1 $\vert$ 0, $\underline{7}$, 35, 203, 1099, 6105, 33797,	R. p. 4
13	1, 1, − 3 $\vert$ 0, $\underline{26}$, 104, 637, 3809, 22074,	R. p. 16
19	1, 1, − 6, − 7 $\vert$ 0, $\underline{38}$, 323, 2209, 13756,	R. p. 26
31	1, 1, −12, −21, 1, 5 $\vert$ $\underline{31}$, 527, 4464,	R. p. 44
37	1, 1, −15, −28, 15, 38, − 1 $\vert$ $\underline{185}$, 2257,	R. p. 52
43	1, 1, −18, −35, 38, 104, 7, − $\underline{6}$, $\vert$ 989,	R. p. 66
61	1, 1, −27, −56, 161, 500, 1, −1023, − 916,	R. p. 92
67	1, 1, −30, −63, 220, 698, −101, −1960, −1758,	R. p. 102
73	1, 1, −33, −70, 288, 929, −298, −3421, −2921,	R. p. 122
79	1, 1, −36, −77, 365, 1193, −617, −5541, −4414,	R. p. 135
97	1, 1, −45, −98, 650, 2183, −2576, −17205, −9748,	

IV. $p = 43$ $\vert$ $1-43\,(y^7+22y^8+\ldots)$,

61 $\vert$ $1-61\,(14y^9+210y^{10}+1062y^{11}+\ldots)$,

67 $\vert$ $1-67\,(4y^9+93y^{10}+1260y^{11}+\ldots)$,

73 $\vert$ $1-73\,(y^9+\ldots)$,

79 $\vert$ $1-79\,(12y^{10}+\ldots)$,

97 $\vert$ $1-97\,(15y^{11}+\ldots)$.

$f = 7.$

I., II.

$$\begin{array}{rll}
1, & -720e+23412 & = -\epsilon+23412, \\
1, & -720e+146865 & = -\epsilon+146845, \\
4, & -2880e+930385 & = -4\epsilon+\ldots, \\
20, & -14400e+5955040 & = -20\epsilon+\ldots, \\
110, & -79200e+38439040 & = -110\epsilon+\ldots, \\
638, & -459300e+249861680 & = -638\epsilon+\ldots, \\
3828, & -2756160e+1633746320 & = -3828\epsilon+\ldots.
\end{array}$$

$(\epsilon = 6!\, e).$

p		
III. $p = 29$	1, 1, 4, 20, $\underline{110}$ $\vert$	R. p. 36
43	1, 1, 4, $\underline{20}$, 110, 638, 3828 $\vert$	R. p. 67
71	1, 1, 4, 20, $\underline{110}$, 638, 3828, 16212, 139645, 901585, 5811040 $\vert$	R. p. 112

IV. $p = 29$ $\vert$ $1-29\,(3y^4+19y^5+\ldots)$,

43 $\vert$ $1-43\,(y^3+12y^5+26y^6+230y^7+\ldots)$,

71 $\vert$ $1-71\,(3y^4+4y^5+35y^6+135y^7+1245y^8+7916y^9+48363y^{10}+\ldots)$

$f=8.$

$$\begin{array}{lll}
\text{I.} & 1, & 8e^2-58e+152, \\
 & 1, & 8e^2-142e+1034, \\
 & -(4e-4), & -\tfrac{1}{3}(32e^3-600e^2+4888e-20943). \\
 & -(4e-25), &
\end{array}$$

$$\epsilon = 4e.$$

$$\begin{array}{lll}
\text{II.} & 1, & \dfrac{1}{2}(\epsilon^2-29\epsilon+304), \\
 & 1, & \dfrac{1}{2}(\epsilon^2-71\epsilon+2068), \\
 & -(\epsilon-4), & -\dfrac{1}{3!}(\epsilon^3-75\epsilon^2+2444\epsilon-41886), \\
 & -(\epsilon-25), &
\end{array}$$

$$\begin{array}{rr|rrrrrrr|rl}
\text{III. } p= & 17 & 1, & 1, & -\;4 & \multicolumn{4}{|l}{\underline{17},} & & \text{R. p. 20} \\
 & 41 & 1, & 1, & -16, & 5, & \underline{62}, & 524 & \multicolumn{2}{|l}{2501,} & \text{R. p. 60} \\
 & 73 & 1, & 1, & -32, & -11, & 278, & \underline{404}, & 741, & & \text{R. p. 123} \\
 & 89 & 1, & 1, & -40, & -19, & 482, & \underline{440}, & -\;939, & & \text{R. p. 149} \\
 & 97 & 1, & 1, & -44, & -23, & 608, & \underline{482}, & -2203, & & \text{R. p. 162}
\end{array}$$

$$\begin{array}{rr|l}
\text{IV. } p= & 41 & 1-41\,(y^4+12y^5+65y^6+\ldots), \\
 & 73 & 1-73\,(6y^5+10y^6+\ldots).
\end{array}$$

$f= 9.$

$$\begin{array}{llll}
\text{I.} & 1, & -(6e-31), & 18e^2-591e+12510, \\
 & 1, & -(6e-221), & 18e^2-1731e+98618, \\
 & 5, & -(30e-1637), & 90e^2-11847e+789617,
\end{array}$$

$$\epsilon = 6e.$$

$$\begin{array}{llll}
\text{II.} & 1, & -(\epsilon-31), & \tfrac{1}{2}(\epsilon^2-197\epsilon+25020), \\
 & 1, & -(\epsilon-221), & \tfrac{1}{2}(\epsilon^2-577\epsilon+197236), \\
 & 5, & -(5\epsilon-1637), & \tfrac{1}{2}(5\epsilon^2-3949\epsilon-1578334).
\end{array}$$

$$\begin{array}{rr|rrrrrrl}
\text{III. } p= & 19 & 1, & 1, & 5 & \multicolumn{3}{|l}{\underline{19},} & \text{R. p. 27} \\
 & 37 & 1, & 1, & 5 & 7, & \underline{197} & \multicolumn{1}{|l}{} & \text{R. p. 53} \\
 & 73 & 1, & 1, & 5, & -17, & \underline{173}, & 1\dot{3}97, & \text{R. p. 124}
\end{array}$$

On the Generalised Equations of Elasticity, and their Application to the Wave Theory of Light. By Prof. K. Pearson, M.A.

[*Read April 11th*, 1889.]

1. In some recent researches with regard to intermolecular action, in which I have endeavoured to explain cohesion by the apparent forces introduced between molecules by certain terms in the usually disregarded kinetic energy of the ether, I have been forced to the conclusion that, on the assumptions of my theory, intermolecular action depends, not only on intermolecular position and aspect, but also on the internal vibratory velocities, and the external translatory velocities of the molecule. The important part played by the former has been recognised in the theory of elasticity by the introduction of the thermal terms. The direct influence of the translatory velocity of the molecule, as a whole, on the law of intermolecular action, and so on the strain-energy, has, I believe, not been hitherto sufficiently regarded. It will probably sensibly disappear from the equations of statical elasticity, but there seems to me every reason for supposing it will have considerable influence in the cases of elastic bodies whose parts are in rapid motion; such cases may possibly arise when we are dealing with problems of impulse and impact of elastic bodies, of sonorous vibrations, but more especially of optical vibrations of elastic media. It is just in these cases of rapid motion that we find the ordinary theory of elasticity goes astray. I need only instance the divergence found by some physicists between the elastic constants as determined by statical and by kinetic methods, the failure of the theory of the longitudinal impact of rods to account for observed facts, and the difficulties which the "perfect elastician" encounters in the etherial regions.* It becomes of interest, therefore, to investigate whether the introduction of the translatory velocity into intermolecular action will account for any of the observed discrepancies. In the particular theory of molecules with which I have occupied myself, the kinetic energy of the ether is expressible as a function of intermolecular distance, aspect, and velocity. This leads to intermolecular action as the outcome of this disregarded energy being a function of

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* See *Hist. of Elasticity*, Vol. I., Art. 1297; Vol. II., Arts. 210, 214; and Glazebrook's *Report on Optical Theories*, British Association, 1886, p. 170 *et seq.*

molecular velocity, and therefore to the action between one element of matter and a second (or one group of molecules and a second), being a function of mean element-velocity and its fluxions with regard to space. But it follows that the strain-energy must in this case be also a function of the velocity and space-fluxions of the velocity of the element.

Let u, v, w be the shifts at any point of an elastic solid; let s_x, s_y, s_z be the three stretches in three rectangular directions x, y, z, and σ_{yz}, σ_{zx}, σ_{xy} the corresponding slides. Let W be the strain-energy *per unit volume* of the solid at x, y, z. Then it has usually been assumed that

$$W = \text{function of } (s_x, s_y, s_z, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}).$$

The conclusions of the theory referred to above indicate that, for an elastic body whose parts are in motion relative to the ether, W ought to be considered as a function also of $\dot{u}$, $\dot{v}$, $\dot{w}$, $\dot{s}_x$, $\dot{s}_y$, $\dot{s}_z$, $\dot{\sigma}_{yz}$, $\dot{\sigma}_{zx}$, $\dot{\sigma}_{xy}$, and $\dot{\tau}_{yz}$, $\dot{\tau}_{zx}$, $\dot{\tau}_{xy}$, where a dot denotes a time-fluxion, and τ_{yz}, τ_{zx}, τ_{xy}* are the twists of the elastic solid at x, y, z, and are here introduced for the purpose of admitting *all* the space-fluxions of the speed components.

Thus, without any further appeal to the particular molecular hypothesis referred to above, we shall suppose W to be of the form

$$W = \phi\,(s_x, s_y, s_z, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}, \dot{u}, \dot{v}, \dot{w}, \dot{s}_x, \dot{s}_y, \dot{s}_z, \dot{\sigma}_{yz}, \dot{\sigma}_{zx}, \dot{\sigma}_{xy}, \dot{\tau}_{yz}, \dot{\tau}_{zx}, \dot{\tau}_{xy}) \quad \text{............(i.)}$$

for any *elastic medium* whatever.

The equations obtained for the shifts u, v, w, on this supposition, I term *the generalised equations of elasticity*.

The form (i.) for W above seems an extremely probable one when we assume—

(*a*) That intermolecular action arises from ether-stress on the molecules.

(*b*) The total kinetic energy of ether and molecules is capable of expression in terms of the positional and speed coordinates of the molecules themselves.

It reduces to the ordinary form in the case of statical elasticity.

---

* $\tau_{yz} = \frac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right)$, $\tau_{zx} = \frac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right)$, $\tau_{xy} = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)$,

if the shift-fluxions are supposed small, and

$$\tau_{yz} = -\tau_{zy}, \quad \tau_{zx} = -\tau_{xz}, \quad \tau_{xy} = -\tau_{yx}.$$

2. I proceed to determine the generalised equations of elasticity on the assumption that the shift-fluxions with regard to space are so small that their squares may be neglected.

Let $L = T - W + V$, where V is the external force-function, and T is the kinetic energy of the element per unit volume, and let

$$L' = \iiint L\, dx\, dy\, dz,$$

taken over the whole elastic medium; then, by the Hamiltonian principle,

$$\delta \int_{t_1}^{t_2} L'\, dt = 0.$$

Now
$$T = \frac{\rho}{2}(\dot{u}^2 + \dot{v}^2 + \dot{w}^2),$$

if ρ be the density of the medium; hence, to find the elastic equations, we must have

$$\delta \int_{t_1}^{t_2} \iiint \left(\frac{\rho}{2}(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + V - W \right) dx\, dy\, dz\, dt = 0 \quad \text{......(ii.)}.$$

As a first variation, change u to $u + \delta u$ without varying t, then

$$\int_{t_1}^{t_2} \iiint \left(\rho \dot{u} \frac{d\,\delta u}{dt} + \frac{dV}{dx} \delta u - \delta W \right) dx\, dy\, dz\, dt = 0.$$

Integrating the first term with regard to the time, and taking $\delta u = 0$ at t_1 and t_2, we have, if dV/dx be written ρX,

$$\int_{t_1}^{t_2} \iiint \{\rho (X - \ddot{u})\, \delta u - \delta W\}\, dx\, dy\, dz\, dt = 0 \quad \text{............(iii.)}.$$

We must now find the variation in W due to δu; this is obviously, when integrated for the time by parts,

$$\begin{aligned}\int_{t_1}^{t_2} \delta W dt = \int_{t_1}^{t_2} \Big\{ &\left(\frac{d\phi}{ds_x} - \frac{d}{dt} \frac{d\phi}{d\dot{s}_x} \right) \delta s_x + \left(\frac{d\phi}{d\sigma_{zx}} - \frac{d}{dt} \frac{d\phi}{d\dot{\sigma}_{zx}} \right) \delta \sigma_{zx} \\ &+ \left(\frac{d\phi}{d\sigma_{xy}} - \frac{d}{dt} \frac{d\phi}{d\dot{\sigma}_{xy}} \right) \delta\sigma_{xy} - \frac{d}{dt} \left(\frac{d\phi}{d\dot{u}} \right) \delta u \\ &- \frac{d}{dt} \left(\frac{d\phi}{d\dot{\tau}_{zx}} \right) \delta\tau_{zx} - \frac{d}{dt} \left(\frac{d\phi}{d\dot{\tau}_{xy}} \right) \delta\tau_{xy} \Big\} dt \quad \text{...............(iv.)}.\end{aligned}$$

Now,
$$\delta s_x = \frac{d\,\delta u}{dx}, \quad \delta\sigma_{zx} = \frac{d\,\delta u}{dz}, \quad \delta\sigma_{xy} = \frac{d\,\delta u}{dy},$$

$$\delta\tau_{xy} = -\frac{1}{2} \frac{d\,\delta u}{dy}, \quad \delta\tau_{zx} = \frac{1}{2} \frac{d\,\delta u}{dz}, \quad \text{.................(v.)}.$$

Further, let x_1, y_1, z_1 be the real coordinates of the point, or

$$x_1 = x+u, \quad y_1 = y+v, \quad z_1 = z+w;$$

then, as in C. Neumann's memoir of 1857 (*Crelle*, Bd. 57, p. 286, or *Hist. of Elast.*, Vol. II., Chap. XI., Sect. I.), it would be correct to change from x, y, z to x_1, y_1, z_1 in the element $dx\,dy\,dz$ of volume, and in the above fluxions of δu. The proper values for this change are given by Saint-Venant, *Leçons de Navier*, p. 791. But, if we are going to neglect the squares of the shift-fluxions, this transformation introduces no alteration into the above forms. Substituting the values (v.) in (iv.), and (iv.) in (iii.), we have, after integration by parts, the following equation:—

$$\begin{aligned}
\int_{t_1}^{t_2}\iiint dx\,dy\,dz\,dt\,\delta u \Big\{ &\Big(\rho\,(X-\ddot{u}) + \frac{d}{dt}\frac{d\phi}{d\dot{u}}\Big) + \frac{d}{dx}\Big(\frac{d\phi}{ds_x} - \frac{d}{dt}\frac{d\phi}{d\dot{s}_x}\Big) \\
&+ \frac{d}{dz}\Big(\frac{d\phi}{d\sigma_{zx}} - \frac{d}{dt}\frac{d\phi}{d\dot{\sigma}_{zx}} - \frac{1}{2}\frac{d}{dt}\frac{d\phi}{d\dot{\tau}_{zx}}\Big) \\
&+ \frac{d}{dy}\Big(\frac{d\phi}{d\sigma_{xy}} - \frac{d}{dt}\frac{d\phi}{d\dot{\sigma}_{xy}} + \frac{1}{2}\frac{d}{dt}\frac{d\phi}{d\dot{\tau}_{xy}}\Big)\Big\} \\
- \int_{t_1}^{t_2}\iint dt\,\delta u \Big\{ &\Big(\frac{d\phi}{ds_x} - \frac{d}{dt}\frac{d\phi}{d\dot{s}_x}\Big)\,dy\,dz \\
&+ \Big(\frac{d\phi}{d\sigma_{xy}} - \frac{d}{dt}\frac{d\phi}{d\dot{\sigma}_{xy}} + \frac{1}{2}\frac{d}{dt}\frac{d\phi}{d\dot{\tau}_{xy}}\Big)\,dz\,dx \\
&+ \Big(\frac{d\phi}{d\sigma_{zx}} - \frac{d}{dt}\frac{d\phi}{d\dot{\sigma}_{zx}} - \frac{1}{2}\frac{d}{dt}\frac{d\phi}{d\dot{\tau}_{zx}}\Big)\,dx\,dy\Big\} = 0 \quad \text{(vi.)}.
\end{aligned}$$

Since δu is arbitrary, we have:—

(i.) For the type of body-shift equation,

$$\begin{aligned}
\rho\,(X-\ddot{u}) &+ \frac{d}{dx}\frac{d\phi}{ds_x} + \frac{d}{dy}\frac{d\phi}{d\sigma_{xy}} + \frac{d}{dz}\frac{d\phi}{d\sigma_{zx}} \\
&+ \frac{d}{dt}\frac{d\phi}{d\dot{u}} - \frac{d}{dt}\Big(\frac{d}{dx}\frac{d\phi}{d\dot{s}_x} + \frac{d}{dy}\frac{d\phi}{d\dot{\sigma}_{xy}} + \frac{d}{dz}\frac{d\phi}{d\dot{\sigma}_{zx}}\Big) \\
&+ \frac{1}{2}\frac{d}{dt}\Big(\frac{d}{dy}\frac{d\phi}{d\dot{\tau}_{xy}} - \frac{d}{dz}\frac{d\phi}{d\dot{\tau}_{zx}}\Big) = 0 \quad \text{(vii.)}.
\end{aligned}$$

(ii.) For the type of surface shift-equation, if l, m, n be the

direction-cosines of the normal to the surface at x, y, z,

$$
\begin{aligned}
& l\frac{d\phi}{ds_x}+m\frac{d\phi}{d\sigma_{xy}}+n\frac{d\phi}{d\sigma_{zx}} \\
& -l\frac{d}{dt}\frac{d\phi}{d\dot{s}_x}-m\frac{d}{dt}\frac{d\phi}{d\dot{\sigma}_{xy}}-n\frac{d}{dt}\frac{d\phi}{d\dot{\sigma}_{zx}} \\
& +\frac{1}{2}\left(m\frac{d}{dt}\frac{d\phi}{d\dot{\tau}_{xy}}-n\frac{d}{dt}\frac{d\phi}{d\dot{\tau}_{zx}}\right)=0 \quad \text{..............(viii.).}
\end{aligned}
$$

If there is a surface load, the right-hand side of the last equation must be replaced by X_0, the component parallel to the axis of x of this load. The first lines of equations (vii.) and (viii.) give the usual equations of elasticity.

Let us write for brevity

$$
\left.
\begin{aligned}
\widehat{xx} &= \frac{d\phi}{ds_x}-\frac{d}{dt}\frac{d\phi}{d\dot{s}_x} \\
\widehat{xy} &= \frac{d\phi}{d\sigma_{xy}}-\frac{d}{dt}\frac{d\phi}{d\dot{\sigma}_{xy}}+\frac{1}{2}\frac{d}{dt}\frac{d\phi}{d\dot{\tau}_{xy}} \\
\widehat{xz} &= \frac{d\phi}{d\sigma_{zx}}-\frac{d}{dt}\frac{d\phi}{d\dot{\sigma}_{zx}}-\frac{1}{2}\frac{d}{dt}\frac{d\phi}{d\dot{\tau}_{zx}}
\end{aligned}
\right\} \quad \text{..............(ix.).}
$$

Then the above equations become

$$
\left.
\begin{aligned}
\rho(X-\ddot{u})+\frac{d}{dt}\frac{d\phi}{d\dot{u}}+\frac{d\widehat{xx}}{dx}+\frac{d\widehat{xy}}{dy}+\frac{d\widehat{xz}}{dz}=0 \\
l\widehat{xx}+m\widehat{xy}+n\widehat{xz}=X_0
\end{aligned}
\right\} \quad \text{...........(x.).}
$$

We may now ask whether $\widehat{xx}$, $\widehat{xy}$, $\widehat{xz}$ may be taken for the generalised stresses upon an elementary plane perpendicular to the direction of x at the point x, y, z, *i.e.*, the traction and the two shears? If this be so, then the surface stress-equation [second of (x.)] is the same in its generalised and customary form, while the body stress-equation differs by the introduction of the term of type

$$\frac{d}{dt}\left(\frac{d\phi}{d\dot{u}}\right).$$

We shall see later that the chief effect of this term is to cause an apparent alteration in the mean density ρ of the elastic solid.

In order to consider this point a little more closely, let us write down the values of $\widehat{yx}$ and $\widehat{yz}$, $\widehat{zx}$ and $\widehat{xy}$. We find

$$\left.\begin{aligned}
\widehat{yx} &= \frac{dw}{d\sigma_{xy}} - \frac{d}{dt}\frac{dw}{d\dot{\sigma}_{xy}} - \frac{1}{2}\frac{d}{dt}\frac{dw}{d\dot{\tau}_{xy}}\\
\widehat{yz} &= \frac{dw}{d\sigma_{yz}} - \frac{d}{dt}\frac{dw}{d\dot{\sigma}_{yz}} - \frac{1}{2}\frac{d}{dt}\frac{dw}{d\dot{\tau}_{yz}}\\
\widehat{zx} &= \frac{dw}{d\tau_{zx}} - \frac{d}{dt}\frac{dw}{d\dot{\sigma}_{zx}} - \frac{1}{2}\frac{d}{dt}\frac{dw}{d\dot{\tau}_{zx}}\\
\widehat{xy} &= \frac{dw}{d\sigma_{yz}} - \frac{d}{dt}\frac{dw}{d\dot{\sigma}_{yz}} - \frac{1}{2}\frac{d}{dt}\frac{dw}{d\dot{\tau}_{yz}}
\end{aligned}\right\} \quad \text{(xi.)}$$

By taking the second of equations (x.), and putting l, m, n pair and pair zero, we see that $\widehat{xx}$, $\widehat{xy}$, $\widehat{xz}$ really represent stresses, and are the traction and two shears across any elementary plane perpendicular to x at the bounding surface. But this bounding surface may be really an imaginary surface in the solid. Hence these quantities are real body-stresses. We are now met by the difficulty that by (ix.) and (xi.), we have not generally

$$\widehat{xy} = \widehat{yx}, \quad \widehat{yz} = \widehat{zy}, \quad \text{and} \quad \widehat{zx} = \widehat{xz} \quad \text{(xii.)}.$$

In order that (xii.) may be true, we must have

$$\frac{dw}{d\dot{\tau}_{xy}}, \quad \frac{dw}{d\dot{\tau}_{yz}}, \quad \frac{dw}{d\dot{\tau}_{zx}}$$

independent of t, and therefore of [illegible] τ_{yz}, τ_{zx}, τ_{xy} and the other quantities which we have supposed to vary with the time. If then (xii.) hold, w cannot be more than a linear function of the twist-speeds, and, since this would be reversed in sign if we reversed the twists, we cannot generally suppose it so. Thus we should be compelled to omit terms involving the twist-speeds from w, or to assume that the functions of the slide-speeds which appear [illegible] occur only in w under the forms [illegible]. There is thus some reason for doubting the value of (xi.) when the strain-energy is a function of the speed of the motion and its space-fluxions.

Boussinesq is, I believe, the only writer who has hitherto thrown any doubt on the absolute validity of (xii.) [illegible]. I am not, however, prepared to say that these relations must necessarily hold when the strain-energy depends on strain-speeds. Indeed it would be dangerous

to do so, because such or similar reasoning would induce us to omit the term $\frac{d}{dt}\frac{d\phi}{d\dot{u}}$ from the typical body shift-equation of (x.).

It is better, therefore, till we have seen whether any physical phenomena require the introduction of the twist-speeds into ϕ, to divide our bodies into two classes:

Class (i.)—Strain-energy a function of twist-speeds.

Types of stresses:

$$\left.\begin{aligned}\widehat{xx} &= \frac{d\phi}{ds_x} - \frac{d}{dt}\frac{d\phi}{d\dot{s}_x} \\ \widehat{yz} &= \frac{d\phi}{d\sigma_{yz}} - \frac{d}{dt}\frac{d\phi}{d\dot{\sigma}_{yz}} + \frac{1}{2}\frac{d}{dt}\frac{d\phi}{d\dot{\tau}_{yz}}\end{aligned}\right\} \text{............ (xiii}a\text{.)}.$$

The other stresses to be found by cyclical interchange, remembering that

$$\dot{\tau}_{yz} = -\dot{\tau}_{zy}.$$

Class (ii.)—Strain-energy does not contain twist-speeds.

Types of stresses:

$$\left.\begin{aligned}\widehat{xx} &= \frac{d\phi}{ds_x} - \frac{d}{dt}\frac{d\phi}{d\dot{s}_x} \\ \widehat{yz} &= \frac{d\phi}{d\sigma_{yz}} - \frac{d}{dt}\frac{d\sigma}{d\dot{\sigma}_{yz}}\end{aligned}\right\} \text{.................... (xiii}b\text{.)}.$$

In both cases, equations (x.) are the stress-equations.

3. I now pass to an analysis of the possible form of ϕ. I shall commence by assuming that ϕ contains no higher powers of the shift-fluxion-speeds than the *second*. This is not so arbitrary as might at first sight appear. Without assuming that ϕ can by Taylor's theorem be expanded in powers of these quantities, and that we need only retain these powers so far as the second, it is sufficient to remark that we are led to believe that the strain-energy contains the shift-fluxion speeds, because these occur in the kinetic energy of the ether, which leads again to their appearance in the apparent intermolecular force, and so in the strain-energy; but it is extremely probable that they appear only in the second powers in the kinetic energy of the ether, and so only as second powers in the apparent intermolecular force, and in

the strain-energy.* I shall adopt, then, the following form for ϕ—

$$\phi = \phi_1 + \phi_2 + \phi_2' + \phi_3 + \phi_3' + \phi_4 + \phi_4' + \phi_{12} + \phi_{13} + \phi_{14} + \phi_{23} + \phi_{24} + \phi_{34},$$

where

$\phi_1 =$ Green's form of the strain-energy, namely, a quadratic function of the strain components.

$\phi_2 =$ quadratic terms in the shift-speeds $\dot{u}, \dot{v}, \dot{w}$.

$\phi_2' =$ linear terms in the shift-speeds „

$\phi_3 =$ quadratic terms in the strain-speeds $\dot{s}_x, \dot{s}_y, \dot{s}_z, \dot{\sigma}_{yz}, \dot{\sigma}_{zx}, \dot{\sigma}_{xy}$.

$\phi_3' =$ linear terms in the strain-speeds „ „

$\phi_4 =$ quadratic terms in the twist-speeds $\dot{\tau}_{yz}, \dot{\tau}_{zx}, \dot{\tau}_{xy}$.

$\phi_4' =$ linear terms in the twist-speeds „

$\phi_{rs} =$ products of r and s quantities, where

for r or $s =$	1	we must take	strains,
„	2	„	shift-speeds,
„	3	„	strain-speeds,
„	4	„	twist-speeds.

I will now consider which of these elements of ϕ may or can exist.

(*a*) ϕ_2' cannot appear; for, starting an unstrained body from rest, the work done would depend not only on the direction, but on the *sense* of the motion.

(*b*) ϕ_{12} seems also improbable; for, by starting a strained body from rest, we could make the work depend on the *sense* of the motion. Possibly, however, this term might appear when magnetisation or other physical influence has given a sense to the ether-field through which the body is moved.

(*c*) and (*d*) ϕ_3' cannot appear; for its sign would be changed by reversing strain-speeds; this applies also to ϕ_4'. But, further, if these terms had any existence in ϕ, they would disappear from the stresses.

(*e*) Assume the system of shifts which connotes the motion of the

* This is certainly the case in molecular researches based on the assumption that the ultimate atom is a pulsating body in a fluid-ether, or again that it is an ether-squirt, or indeed a body of any shape moving through a liquid; see Lamb's "Motion of Fluids," § 110 *et seq.*

body as a whole:

$$u = -\omega_3 y + \omega_2 z + u_0,$$
$$v = -\omega_1 z + \omega_3 x + v_0,$$
$$w = -\omega_2 x + \omega_1 y + w_0,$$

where u_0, v_0, w_0, ω_1, ω_2, ω_3 are functions of t. Then $\dot{u}_0$, $\dot{v}_0$, $\dot{w}_0$ are the velocity components, and $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$ the spin components of the elastic solid as a rigid whole. We find all the strains and strain-speeds zero, and $\dot{r}_{yz} = \dot{\omega}_1$, $\dot{r}_{zx} = \dot{\omega}_2$, $\dot{r}_{xy} = \dot{\omega}_3$. Let us now inquire what terms can occur in ϕ_{24}. Give the body a twist on a screw round the axis of x, only $\dot{u}_0$ and $\dot{\omega}_1$ remain. Their product would denote a change in sign of work according as the screw was right- or left-handed. This would be impossible for an elastic body in a perfectly isotropic medium. If any physical field should somehow change this perfect isotropy, so that to rotate three axes as a whole does not affect the value of ϕ, but to *reverse only one of them* does, then such terms as ϕ_{24} may occur. We will term such media with Boussinesq *dissymmetrically isotropic*. In a perfectly isotropic medium, not only can no terms like $\dot{u}_0\,\dot{\omega}_1$, but no terms like $\dot{u}_0\,\dot{\omega}_2$ can occur. For it cannot matter whether a body, spinning about an axis perpendicular to its direction of translation, has a positive or negative spin. Hence for such a body $\phi_{24} = 0$ Generally, we shall suppose $\phi_{24} = 0$.

(*f*) ϕ_{14} cannot exist for an elastic body in a perfectly isotropic ether, for the *sense* in which a strained body is rotated as a whole cannot affect the value of ϕ. Generally, we shall suppose $\phi_{14} = 0$.

(*g*) Suppose a body to receive a slight compression parallel to the x-axis, and to have a velocity of translation $\dot{u}_0$ in the same direction, so that we may take $u = -ax + u_0$, and thus $\dot{u} = -\dot{a}x + \dot{u}_0$, $\dot{s}_x = -\dot{a}$. Thus $\dot{u}\dot{s}_x$, if $\dot{a}$ small as compared with $\dot{u}_0$, would change its sign with $\dot{u}_0$, and this does not seem possible for the strain-energy of an elastic body in a perfectly isotropic ether. Still less can the sense of velocities perpendicular to the direction of compression affect the work done, or terms of the type $\dot{u}\dot{s}_y$ cannot appear. Similar reasoning shows the impossibility of terms like $\dot{u}\dot{\sigma}_{xy}$ and $\dot{u}\dot{\sigma}_{xz}$, for the work done by a slide-speed perpendicular to x cannot be affected by a velocity of the body as a whole parallel to x. Nor is $\dot{u}\dot{\sigma}_{yz}$ any more admissible than these, and for like reasons. Hence we see that ϕ_{23} can contribute nothing to ϕ.

(*h*) Finally, let us inquire as to the probability of the terms in ϕ_{34}. First, no terms of the types $\dot{\tau}_{yz}\,\dot{s}_x$, or $\dot{\tau}_{yz}\,\dot{s}_y$ are likely to occur; for, if they did, then a simple compression, accompanied by a twist round its direction or perpendicular to its direction, would do work having its sign changed with the sign of the spin, which is obviously improbable. Further, no terms of type $\dot{\tau}_{yz}\,\dot{\sigma}_{xz}$ seem likely to occur, for the work done by the slide-speed of a face perpendicular to x parallel to z, combined with a spin round x (*e.g.*, take $w = bx - cy$, $v = cz$), can hardly be affected by a change in sign of twist-speed (*i.e.*, in $\dot{c}$). There remain terms of the form $\dot{\tau}_{yz}\,\dot{\sigma}_{yz}$. Combine a rotation round x with a slide of a face perpendicular to z parallel to y, or take

$$v = -\omega z + cz, \quad w = \omega y + cy.$$

$\dot{\tau}_{yz}\,\dot{\sigma}_{yz} = 2\dot{\omega}\dot{c}$, and so changes its sign with the spin $\dot{\omega}$, and this again leads to an improbable result. Thus we should not expect terms of the form ϕ_{34} to occur.

The conclusion we thus draw is, that while no motion of the elastic body as a rigid whole will affect the work done by the strains, still such motion contributes its own quota to the strain-energy. Further, the strain-speeds do alter the work due to a system of strains, or the strain-energy depends upon the rate at which the strain is made. This result will not appear to contradict the principle of energy, if we only remember that the ether is perpetually acting as a conductor of energy, and that we ought to apply the principle of energy not only to the elastic body, but to the movement of the ether in and about the body.

4. We have found in the previous article that we must have ϕ of the form

$$\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_{12} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{xiv.}).$$

I now proceed to inquire into th[illegible] of the terms wh[illegible]h can arise in these several parts of ϕ.

(*a*) Of ϕ_1 it is needless to spe[illegible] contains tw[illegible] dependent constants, for which I [illegible] present t[illegible] of the *History of Elasticity*, Vol. I. [illegible] Vol. II., p. [illegible]

(*b*) ϕ_2 may be taken of the for[illegible]

$$[illegible] = \tfrac{1}{2}(\kappa_{11}\dot{u}^2 + \kappa_{22}\dot{v}^2 + \kappa_{33}\dot{w}^2 \ldots [illegible] \ldots + 2\kappa_{23}\dot{u} \ldots$$

Can [illegible] occur? Suppos[illegible] out strai[illegible]

moving through an isotropic ether; then, to reverse *two* of the velocity components would alter the amount of work done; but there seems no reason why this should not be true, provided x, y, z are not axes of elastic symmetry for the body; if they are, to change one component speed ought not to alter work done; thus in this case, and this only, we are justified in putting

$$\kappa_{23} = \kappa_{13} = \kappa_{31} = 0.$$

Otherwise we must retain the full value of ϕ_2.

For an elastic body with three rectangular axes of elastic symmetry

$$\phi_2 = \tfrac{1}{2}(\kappa_{11}\dot{u}^2 + \kappa_{22}\dot{v}^2 + \kappa_{33}\dot{w}^2).$$

For an isotropic body

$$\phi_2 = \tfrac{1}{2}\kappa(\dot{u}^2 + \dot{v}^2 + \dot{w}^2).$$

(*c*) In general, ϕ_3 will contain twenty-one independent constants. We shall represent them by the same symbols as the constants of Green's function or the strain-energy, with a dash or subscript affixed. We must investigate how these will be reduced (i.) when there are three axes of elastic symmetry, (ii.) when there is complete isotropy.

Now the reduction of ϕ_3 follows almost the same lines as that for ϕ_1. Changes of the type,

$$-x, \ -\dot{u}, \ -\dot{\sigma}_{xz}, \ -\dot{\sigma}_{xy}, \ \text{for} \ x, \ \dot{u}, \ \dot{\sigma}_{xz}, \ \dot{\sigma}_{xy},$$

cannot affect the value of ϕ_3 if there be three planes of elastic symmetry. Nor can rotations round the intersections of these planes affect the value, if there be isotropy. Hence we find for—

(i.) Three rectangular axes,

$$\phi_3 = \tfrac{1}{2}\{a_1\dot{s}_x^2 + b_1\dot{s}_y^2 + c_1\dot{s}_z^2 + 2d_1'\dot{s}_y\dot{s}_z + 2e_1'\dot{s}_z\dot{s}_x + 2f_1'\dot{s}_x\dot{s}_y$$
$$+ d_1\dot{\sigma}_{yz}^2 + e_1\dot{\sigma}_{zx} + f_1\dot{\sigma}_{xy}^2\},$$

where a_1, b_1, c_1, &c. denote elastic constants.

(ii.) Complete isotropy,

$$\phi_3 = \tfrac{1}{2}\{\lambda'\dot{\theta}^2 + 2\mu'(\dot{s}_x + \dot{s}_y^2 + \dot{s}_z) + \mu'(\dot{\sigma}_{yz}^2 + \dot{\sigma}^{zx} + \dot{\sigma}_{xy}^2)\},$$

where $$\dot{\theta} = \dot{s}_x + \dot{s}_y + \dot{s}_z = \frac{d}{dt}(s_x + s_y + s_z).$$

(*d*) We now pass to ϕ_4. Its most general form is given by

$$\tfrac{1}{2}\{\gamma_{11}\dot{\tau}_{yz}^2 + \gamma_{22}\dot{\tau}^{zx} + \gamma_{33}\dot{\tau}_{xy}^2 + 2\gamma_{23}\dot{\tau}_{zx}\dot{\tau}_{xy} + 2\gamma_{31}\dot{\tau}_{yz}\dot{\tau}_{xy} + 2\gamma_{12}\dot{\tau}_{yz}\dot{\tau}_{zx}\}.$$

There seems no reason why all these terms should not occur in the most general æolotropic solid. But we obviously have for—

(i.) Three rectangular axes,

$$\phi_4 = \tfrac{1}{2}\left\{\gamma_{11}\dot{\sigma}_{yz}^2 + \gamma_{22}\dot{\sigma}_{zx}^2 + \gamma_{33}\dot{\sigma}_{xy}^2\right\}.$$

(ii.) Complete isotropy,

$$\phi_4 = \tfrac{1}{2}\gamma\left\{\dot{\sigma}_{yz}^2 + \dot{\sigma}_{zx}^2 + \dot{\sigma}_{xy}^2\right\}.$$

(*e*) It remains to consider ϕ_{13} which contains the product of the component strains, and their first time-fluxions. Its most general value contains thirty-six constants, and we must investigate whether all these terms are possible. In the most general case of æolotropy, I do not see that it is necessary for any of these coefficients to vanish. We may then write

$$\begin{aligned}
\phi_{13} = {} & a_{11}s_x\dot{s}_x + a_{12}s_x\dot{s}_y + a_{13}s_x\dot{s}_z + a_{14}s_x\dot{\sigma}_{yz} + a_{15}s_x\dot{\sigma}_{zx} + a_{16}s_x\dot{\sigma}_{xy} \\
& + a_{21}s_y\dot{s}_x + a_{22}s_y\dot{s}_y + a_{23}s_y\dot{s}_z + a_{24}s_y\dot{\sigma}_{yz} + a_{25}s_y\dot{\sigma}_{zx} + a_{26}s_y\dot{\sigma}_{xy} \\
& + a_{31}s_z\dot{s}_x + a_{32}s_z\dot{s}_y + a_{33}s_z\dot{s}_z - a_{34}s_z\dot{\sigma}_{yz} + a_{35}s_z\dot{\sigma}_{zx} + a_{36}s_z\dot{\sigma}_{xy} \\
& + a_{41}\sigma_{yz}\dot{s}_x + a_{42}\sigma_{yz}\dot{s}_y + a_{43}\sigma_{yz}\dot{s}_z + a_{44}\sigma_{yz}\dot{\sigma}_{yz} + a_{45}\sigma_{yz}\dot{\sigma}_{zx} + a_{46}\sigma_{yz}\dot{\sigma}_{xy} \\
& + a_{51}\sigma_{zx}\dot{s}_x + a_{52}\sigma_{zx}\dot{s}_y + a_{53}\sigma_{zx}\dot{s}_z + a_{54}\sigma_{zx}\dot{\sigma}_{yz} - a_{55}\sigma_{zx}\dot{\sigma}_{zx} + a_{56}\sigma_{zx}\dot{\sigma}_{xy} \\
& + a_{61}\sigma_{xy}\dot{s}_x + a_{62}\sigma_{xy}\dot{s}_y + a_{63}\sigma_{xy}\dot{s}_z + a_{64}\sigma_{xy}\dot{\sigma}_{yz} + a_{65}\sigma_{xy}\dot{\sigma}_{zx} + a_{66}\sigma_{xy}\dot{\sigma}_{xy}.
\end{aligned}$$

Let us now inquire how many of these terms occur when there are three planes of elastic symmetry. Now the value of ϕ_{13} must remain unchanged, if

$$-x,\ -u,\ -\dot{u},\ -\sigma_{yx},\ -\dot{\sigma}_{zx},\ -\sigma_{xy},\ -\dot{\sigma}_{xy}$$

are written for the like plus quantities: and for two other interchanges of the like type. This frees ϕ_{14} from all products of the form $s\dot{\sigma}$ or $\dot{s}\sigma$, as well as all products of unlike slide and slide-speed, and so reduces the constants to 12. I do not see that this number can be further reduced, although I am inclined to suppose that $a_{pq} = a_{qp}$, even in this case.

For complete isotropy we must have

$$a_{11} = a_{22} = a_{33}, \quad a_{44} = a_{55} = a_{66},$$

$$a_{12} = a_{13} = a_{21} = a_{23} = a_{32} = a_{31}.$$

Thus we can at once write ϕ_{13} in the form

$$\phi_{13} = \frac{1}{2}\frac{d}{dt}\left\{a_0\left(s_x^2+s_y^2+s_z^2\right)+2b_0\left(s_y s_z+s_z s_x+s_x s_y\right)+c_0\left(\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2\right)\right\}.$$

But to the quantity in curled brackets we ought to be able to apply a rotation round the axis of z without altering its form. This involves, exactly as in the case of ϕ_1, our being able to throw it into the form

$$\phi_{13} = \frac{1}{2}\frac{d}{dt}\left\{\lambda''\theta^2+2\mu''\left(s_x^2+s_y^2+s_z^2\right)+\mu''\left(\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2\right)\right\}.$$

5. We can now draw some general conclusions with regard to the number of elastic constants; we have

	Æolotropic Solid.	Three axes of Elastic Symmetry.	Isotropy.
ϕ_1	21	9	2
ϕ_2	6	3	1
ϕ_3	21	9	2
ϕ_4	6	3	1
ϕ_{13}	36 (21?)	12 (9?)	2
ϕ	90 (75?)	36 (33?)	8

Thus the æolotropic elastic solid has 90 independent elastic constants, that with three axes of elasticity 36, and the isotropic elastic solid 8. If equations of the type $\widehat{xz} = \widehat{zx}$ hold, the twist-speeds disappear, or these numbers must be reduced by 6, 3, and 1, respectively. The numbers with a query after them in brackets give the numbers of constants on the hypothesis that ϕ_{13} is the differential with regard to the time of a quadratic function of s_x, s_y, s_z, σ_{yz}, σ_{zx}, σ_{xy}, which seems to me a probable hypothesis. The minimum number of independent constants would thus be, when the twist-speeds disappear, 69, 30, and 7.

It is, of course, quite possible that any special molecular hypothesis may lead to relations between these constants, so that we ought not to be surprised if any particular physical phenomena require for their explanation a relation between some of these constants.

6. To find the generalised equations of elasticity for an isotropic elastic solid.

Here we have

$$
\begin{aligned}
\phi = {} & \tfrac{1}{2}\left\{\lambda\theta^2+2\mu\,(s_x^2+s_y^2+s_z^2)+\mu\,(\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2)\right\} \\
& +\tfrac{1}{2}\kappa\,(\dot{u}^2+\dot{v}^2+\dot{w}^2)+\tfrac{1}{2}\left\{\lambda'\dot{\theta}^2+2\mu'\,(\dot{s}_x^2+\dot{s}_y^2+\dot{s}_z^2)+\mu'\,(\dot{\sigma}_{yz}^2+\dot{\sigma}_{zx}^2+\dot{\sigma}_{xy}^2)\right\} \\
& +\tfrac{1}{2}\gamma\,\left\{\dot{\tau}_{yz}^2+\dot{\tau}_{zx}^2+\dot{\tau}_{xy}^2\right\}+\left\{\lambda''\theta\dot{\theta}+2\mu''\,(s_x\dot{s}_x+s_y\dot{s}_y+s_z\dot{s}_z)\right. \\
& \left.+\mu''(\sigma_{yz}\dot{\sigma}_{yz}+\sigma_{zx}\dot{\sigma}_{zx}+\sigma_{xy}\dot{\sigma}_{xy})\right\} \quad \text{...(xv.)}
\end{aligned}
$$

We can now write down the stresses

$$
\left.
\begin{aligned}
\widehat{xy} &= \lambda\theta+2\mu s_x-\frac{d^2}{dt^2}(\lambda'\theta+2\mu' s_x) \\
\widehat{yy} &= \lambda\theta+2\mu s_y-\frac{d^2}{dt^2}(\lambda'\theta+2\mu' s_y) \\
\widehat{zz} &= \lambda\theta+2\mu s_z-\frac{d^2}{dt^2}(\lambda'\theta+2\mu' s_z) \\
\widehat{yz} &= \mu\sigma_{yz}-\frac{d^2}{dt^2}\left(\mu'\sigma_{yz}-\frac{\gamma}{2}\tau_{yz}\right) \\
\widehat{zx} &= \mu\sigma_{zx}-\frac{d^2}{dt^2}\left(\mu'\sigma_{zx}-\frac{\gamma}{2}\tau_{zx}\right) \\
\widehat{xy} &= \mu\sigma_{xy}-\frac{d^2}{dt^2}\left(\mu'\sigma_{xy}-\frac{\gamma}{2}\tau_{xy}\right) \\
\widehat{zy} &= \mu\sigma_{yz}-\frac{d^2}{dt^2}\left(\mu'\sigma_{yz}+\frac{\gamma}{2}\tau_{yz}\right) \\
\widehat{xz} &= \mu\sigma_{zx}-\frac{d^2}{dt^2}\left(\mu'\sigma_{zx}+\frac{\gamma}{2}\tau_{zx}\right) \\
\widehat{yx} &= \mu\sigma_{xy}-\frac{d^2}{dt^2}\left(\mu'\sigma_{xy}+\frac{\gamma}{2}\tau_{xy}\right)
\end{aligned}
\right\} \text{..................(xvi.).}
$$

If we put $\gamma = 0$, or suppose the strain-energy due to twist-speeds zero, we have

$$\widehat{yz} = \widehat{zy}, \quad \widehat{zx} = \widehat{xz}, \quad \text{and} \quad \widehat{xy} = \widehat{yx},$$

as usual.

If the values (xvi.) are substituted in (x.), we have the body and surface strain-equations; substituting for the strains their values in turns of the shift-fluxions, we have the shift-equations for an isotropic elastic solid.

We note that the terms of ϕ_{13} do not occur in the values of the stresses.* This will invariably be the case when in the value of ϕ_{13} we have relations of the form $a_{pq} = a_{qp}$: see page 308. This I think extremely probable, even in the case of æolotropic bodies. Thus ϕ_{13}, while a sensible part of the strain-energy, will not affect the stresses or the strain-equations.

I propose to apply these equations in the following articles to one or two special cases of strain and vibration.

7. *Elastic After-strain.* — Suppose it possible to apply a single uniform stress throughout a body, and that this be done without producing vibrations, or that these vibrations are so small as to be negligible.

Case (i.) *Uniform Traction.*—Let $\widehat{xx} = T$, and all the other stresses be zero. We have $s_y = s_z$, and:

$$T = 2\lambda s_y + (\lambda+2\mu)\, s_x - \frac{d^2}{dt^2}\left\{2\lambda' s_y + (\lambda'+2\mu')\, s_x\right\},$$

$$0 = \lambda s_x + 2\,(\lambda+\mu)\, s_y - \frac{d^2}{dt^2}\left\{\lambda' s_x + 2\,(\lambda'+\mu')\, s_y\right\}.$$

To solve these equations, assume

$$s_x = A' + Ae^{mt},$$

$$s_y = B' + Be^{mt}.$$

We find
$$B' = -\frac{\lambda}{2\,(\lambda+\mu)}\, A' = -\eta A',$$

$$A' = \frac{\lambda+\mu}{\mu\,(3\lambda+2\mu)}\, T = \frac{T}{E},$$

if η be the stretch-squeeze ratio, and E the stretch-modulus.

Further, we have from the exponential terms in the time:

$$2\,(\lambda-\lambda' m^2)\, B + \left\{(\lambda+2\mu) - m^2\,(\lambda'+2\mu')\right\} A = 0,$$

$$(\lambda-\lambda' m^2)\, A + \left\{2\,(\lambda+\mu) - 2\,(\lambda'+\mu')\, m^2\right\} B = 0;$$

---

* Generally $\left[\dfrac{d}{ds_x} - \dfrac{d}{dt}\,\dfrac{d}{d\dot{s}_x}\right] s_x \dot{s}_x = 0.$

whence we obtain the equation for m^2,

$$(\lambda-\lambda' m^2)^2 = \{(\lambda+\mu)-(\lambda'+\mu')\,m^2\}\{(\lambda+2\mu)-(\lambda'+2\mu')\,m^2\},$$

or $\mu'(3\lambda'+2\mu')\,m^4 - \{\mu'(3\lambda+2\mu)+\mu(3\lambda'+2\mu')\}\,m^2+\mu(3\lambda+2\mu)=0.$

Solving this equation, we find

$$m^2 = \mu/\mu' \text{ and } = (3\lambda+2\mu)/(3\lambda'+2\mu').$$

Now m cannot be *positive*, so long at least as we are dealing with elastic-strain. For λ' and μ' are small as compared with λ and μ, the effects we are considering being only of the second order. Hence m^2 is large, and, if m were positive, the strain would rapidly grow immensely large, which is contrary to experience. Thus we must give m the negative values

$$-\sqrt{\frac{\mu}{\mu'}} \text{ and } -\sqrt{\frac{3\lambda+2\mu}{3\lambda'+2\mu'}}.$$

Thus
$$s_x = \frac{T}{E} + A_1 e^{-\sqrt{(\mu/\mu')}\,t} + A_2 e^{-\sqrt{(3\lambda+2\mu)/(3\lambda'+2\mu')}\,t},$$

$$s_y = -\frac{\eta T}{E} + B_1 e^{-\sqrt{(\mu/\mu')}\,t} + B_2 e^{-\sqrt{(3\lambda+2\mu)/(3\lambda'+2\mu')}\,t}.$$

It remains to determine B_1, B_2 in terms of A_1, A_2, respectively. Substituting for m^2, in the equation connecting A and B above, we readily find

$$B_1 = -\tfrac{1}{2}A_1, \quad \text{and} \quad B_2 = A_2.$$

Thus, finally,
$$\left.\begin{aligned} s_x &= \frac{T}{E} + A_1 e^{-\sqrt{(\mu/\mu')}\,t} + A_2 e^{-\sqrt{(3\lambda+3\mu)/(3\lambda'+2\mu')}\,t} \\ s_y &= -\frac{\eta T}{E} - \frac{A_1}{2} e^{-\sqrt{(\mu/\mu')}\,t} + A_2 e^{-\sqrt{(3\lambda+2\mu)/(3\lambda'+2\mu')}\,t} \end{aligned}\right\} \ldots \text{(xvii.)}.$$

These results seem suggestive, and not without experimental confirmation in the phenomena classed as "elastic after-strain." We cannot, of course, determine A_1 and A_2 without some further hypothesis as to the state of strain when $t = 0$. We see, however, that:

(*a*) Elastic after-stretch in the case of uniform tensile stress consists of *two* exponential terms. These terms have for 'exponential phases' respectively,

$$\sqrt{\frac{\mu}{\mu'}}\,t \quad \text{and} \quad \sqrt{\frac{3\lambda}{\lambda'}}$$

(*b*) Elastic after-dilatation contains only one exponential term, and this with 'exponential phase,'

$$\sqrt{\frac{3\lambda+2\mu}{3\lambda'+2\mu'}}\, t.$$

For, the dilatation

$$\theta = s_x + 2s_y,$$

$$= \frac{T}{3\lambda+2\mu} + 3A_2 e^{-\sqrt{(3\lambda+2\mu)/(3\lambda'+2\mu')}\, t}.$$

This may be compared with the result of a uniform pressure applied all over the surface of an isotropic elastic solid. In this case

$$\widehat{xx} = \widehat{yy} = \widehat{zz} = -p,$$

whence we find, from (xvi.),

$$-3p = (3\lambda+2\mu)\,\theta - \frac{d^2}{dt^2}\left\{(3\lambda'+2\mu')\,\theta\right\}.$$

Thus we must have

$$\theta = -\frac{3p}{3\lambda+2\mu} + Ce^{-\sqrt{(3\lambda+2\mu)/(3\lambda'+2\mu')}\, t} \quad \text{..............(xviii.)}.$$

In the *History of Elasticity*, Vol. I., p. 885, I have termed $\frac{3\lambda+2\mu}{3}$ the dilatation modulus, and represented it by the letter F; if

$$F' = \frac{3\lambda'+2\mu'}{3},$$

we may term it by analogy the ***after-dilatation modulus***, and then find

$$\theta = -\frac{p}{F} + Ce^{-\sqrt{(F/F')}\, t}.$$

Thus one portion of the after-strain in tensile stress is due to the influence of after-dilatation.

Case (ii.) *Uniform Shear.*—Let all the stresses except $\widehat{zy}$ be zero, and let this have the uniform value S at all points of the body. Assume $\gamma = 0$, or the twist-speeds zero, we find

$$S = \mu\sigma_{yz} - \frac{d^2}{dt^2}(\mu'\sigma_{yz}).$$

The suitable solution is in this case

$$\sigma_{yz} = \frac{S}{\mu} + De^{-\sqrt{(\mu/\mu')}\,t} \quad \text{..................... (xix.)}.$$

Thus elastic after-slide is expressed by single exponential in the time, with the exponential phase $\sqrt{\frac{\mu}{\mu'}}\,t$. By analogy to μ the slide-modulus, we may term μ' the *after-slide-modulus*. Thus the square roots of the ratios of the slide- and after-slide-moduli, and of the dilatation- and after-dilatation-moduli, are the constants on which after-strain depends. The after-stretch is only a combination of after-slide and after-dilatation terms.

8. As I have said, the quantities A_1, A_2, C and D of the previous article will depend on the strain at the instant when the uniform stress is applied to the body, and we ought properly to take account of the vibrations excited by the application of a load of given magnitude, which is the nearest attempt we can practically make to the application of a uniform stress to each element of the elastic solid.

If we suppose that s_x, s_y, θ, and σ_{yz} are all zero initially, we easily find

$$\left.\begin{aligned} s_x &= T\left\{\frac{1}{E} - \frac{1}{3\mu}e^{-\sqrt{(\mu/\mu')}\,t} - \frac{1}{9F}e^{-\sqrt{(F/F')}\,t}\right\} \\ s_y &= -T\left\{\frac{\eta}{E} - \frac{1}{6\mu}e^{-\sqrt{(\mu/\mu')}\,t} + \frac{1}{9F}e^{-\sqrt{(F/F')}\,t}\right\} \\ \theta &= -\frac{p}{F}\left\{1 - e^{-\sqrt{(F/F')}\,t}\right\} \\ \sigma_{yz} &= \frac{S}{\mu}\left\{1 - e^{-\sqrt{(\mu/\mu')}\,t}\right\} \end{aligned}\right\} \quad \text{......... (xx.)}.$$

It is obvious that experiments made to determine the stretch-, slide-, or dilatation-moduli will not give the true values (E, μ, and F) of these quantities unless sufficient time has been allowed to elapse so that the exponential terms have become insensible.

9. *On the longitudinal vibrations of an elastic cord or bar of small cross-section.*

In this case, let x be taken as the axis of the rod, and let y, z be any rectangular axes in the plane of the cross-section. Then s_y, s_z are functions only of x and t. Further,

$$\widehat{yy} = \widehat{zz} = 0$$

for the whole rod. It is usual to assume that the equation for the vibrations will be of the form

$$\rho\ddot{u} = \frac{d\,\widehat{xx}}{dx} \quad \text{.............................(xxi.).}$$

This supposes $\frac{d\,\widehat{xy}}{dy}$ and $\frac{d\,\widehat{xz}}{dz}$ zero or negligible, but

$$\frac{d\,\widehat{xy}}{dy} = \mu\frac{d\sigma_{xy}}{dy} = \mu\frac{ds_y}{dx}, \quad \text{and} \quad \frac{d\,\widehat{xz}}{dz} = \mu\frac{ds_z}{dx}.$$

Hence we cannot omit these terms without supposing $\frac{ds_y}{dx}$ and $\frac{ds_z}{dx}$ small or negligible. Now $\widehat{xx}$ in the above equation is usually written $= Es_x$, which of course assumes $s_y = s_z = -\eta s_x$; and, since η only averages about $\frac{1}{4}$, we cannot neglect ds_y/dx and ds_z/dx without neglecting ds_x/dx, which leads to $d\,\widehat{xx}/dx$ being neglected, and therefore to an absurdity. If we do not assume $d\,\widehat{xy}/dy$, $d\,\widehat{xz}/dz$, or $\widehat{xy}$ and $\widehat{xz}$ zero, we find an unequilibrated surface-load on the sides of the rod. Mr. Chree has noted this difficulty in a paper "On the Longitudinal Vibrations of a Circular Bar" (*Quarterly Journal of Mathematics*, Vol. XXI., p. 287), and shown how closely equation (xxi.) gives an approximate result. Assuming that the reasoning on which that equation is based holds in the case we are dealing with, we have the following equations to solve, where D is written for $\frac{d}{dt}$, and we include terms in ϕ_2, but not ϕ_4:—

$$(\rho-\kappa)\,D^2u = \frac{d\,\widehat{xx}}{dx} \quad \text{.......................(xxii.),}$$

$$\widehat{xx} = (\overline{\lambda+2\mu}-\overline{\lambda'+2\mu'}\,D^2)\,s_x + 2\,(\lambda-\lambda'D^2)\,s_y,$$

$$0 = \left[2\,(\lambda+\mu)-2\,(\lambda'+\mu')\,D^2\right]s_y + (\lambda-\lambda'D^2)\,s_x.$$

Eliminating s_y, we find

$$\widehat{xx} = \left[\overline{\lambda+2\mu}-\overline{\lambda'+2\mu'}\,D^2 - \frac{(\lambda-\lambda'D^2)^2}{\lambda+\mu-(\lambda'+\mu')\,D^2}\right]s_x.$$

Now let $u = Ae^{mt+nx}$, and therefore $s_x = nAe^{mt+nx}$. It follows that, if we write

$$m_1^2 = \frac{\mu}{\mu'}, \quad m_2^2 = \frac{3\lambda+2\mu}{3\lambda'+2\mu'}, \quad \text{and} \quad m_3^2 = \frac{\lambda+\mu}{\lambda'+\mu'},$$

where m_1, m_2, m_3 will be, in a certain sense, constants of after-strain, we have

$$\widehat{xx} = nE\frac{\left(1-\frac{m^2}{m_1^2}\right)\left(1-\frac{m^2}{m_2^2}\right)}{1-\frac{m^2}{m_3^2}}$$

Thus, from (xxii.), it follows that

$$(\rho-\kappa)\,m^2 = n^2E\frac{\left(1-\frac{m^2}{m_1^2}\right)\left(1-\frac{m^2}{m_2^2}\right)}{1-\frac{m^2}{m_3^2}} \quad\text{............(xxiii.).}$$

Had we included the twist terms from equation (vii.), we should have had to introduce into the right of (xxii.) the term

$$\frac{\gamma}{2}D^2\left(\frac{d\tau_{xy}}{dy}-\frac{d\tau_{zx}}{dz}\right) = \frac{\gamma}{4}D^2\left(\frac{ds_y}{dx}+\frac{ds_z}{dx}\right) = \frac{\gamma}{2}D^2\frac{ds_y}{dx}.$$

This would have given us an additional term on the right in (xxiii.), namely,

$$-\gamma\eta n^2\frac{m^2\left(1-\frac{m^2}{m_4^2}\right)}{1-\frac{m^2}{m_3^2}},$$

where $$m_4^2 = \lambda/\lambda'.$$

Now m_1, m_2, m_3, and m_4 are large. Hence, to a first approximation,

$$m^2 = n^2\,\frac{E}{\rho-\kappa}.$$

This agrees with the ordinary theory, except that ρ is changed to $\rho-\kappa$, or there is a small apparent change in the density of the material. The above relation denotes that all waves of sound travel longitudinally with the same velocity through the bar. Putting

$$\Omega'^2 = \frac{E}{\rho-\kappa} = \frac{\rho}{\rho-\kappa}\,\Omega_0^2 = \Omega_0^2\left(1+\frac{\kappa}{\rho}\right),$$

we have Ω' = velocity of sound by the first approximation of this theory, and Ω_0 = velocity of sound on the old theory.

Now, let $$m = \sqrt{-1}\,\frac{2\pi}{l}\,\Omega, \quad\text{and}\quad n = \sqrt{-1}\,\frac{2\pi}{l},$$

then we find

$$\Omega^2 = \frac{\Omega'^2 \left(1 + \frac{4\pi^2}{m_1^2 l^2}\Omega^2\right)\left(1 + \frac{4\pi^2}{m_2^2 l^2}\Omega^2\right)}{\left(1 + \frac{4\pi^2}{m_3^2 l^2}\Omega^2\right)} \quad \text{...........(xxiv.)}$$

as the equation to determine Ω. We see at once that the velocity of sound, owing to the after-strain terms, depends on the length of the sound-wave l. Including the twist-term, and retaining only first powers of small quantities, we have

$$\Omega^2 = \Omega_0^2 \left\{ 1 + \frac{\kappa}{\rho} + \frac{4\pi^2}{l^2}\Omega_0^2 \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{1}{m_3^2} + \frac{\gamma\eta}{E} \right) \right\} \quad \text{......(xxv.)}.$$

Now Lord Rayleigh* and Mr. Chree† have shown that for a bar of circular cross-section, if we include the effect of lateral contraction,

$$\Omega^2 = \Omega_0^2 \left\{ 1 - \frac{1}{2}\frac{4\pi^2}{l^2} a^2 \eta^2 \right\},$$

where l is, as before, the wave-length, and a is the radius of the cross-section.

Thus the after-strain influence on the longitudinal velocity of sound, or the correction for lateral motion, will be the more important according to the relative magnitude of $\frac{\Omega_0^2}{m_1^2}$ and $a^2\eta^2$, or according to the relative magnitude of $\frac{1}{m_1}$ and $\frac{a}{\Omega_0}$. Now $\frac{a}{\Omega_0}$ is the time sound would take to traverse a distance equal to the radius of the cross-section, while $1/m_1$ is the time in which the increments of after-strain decrease $1/e$ of themselves. In most cases the latter is a time capable of measurement, hence it would seem that for a long rod of small cross-section the variation in the velocity of sound due to after-strain influence would be of considerably more importance than that due to lateral motion.

10. *On the propagation of waves in an isotropic elastic solid.*‡

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* *Theory of Sound*, Vol. I., § 157.

† In the memoir above referred to, equation (34), p. 295.

‡ The equations obtained for light vibrations in the following sections are kindred in form with those given by Boussinesq in various papers to which reference will be made in the sequel. Boussinesq deduces them from a very different hypothesis, however. Voigt, in Vol. XIX. of *Wiedemann's Annalen*, 1883, has two papers: one, pp. 691–704, a criticism of Ketteler; another, pp. 873–908, entitled: *Theorie des Lichtes für vollkommen durchsichtige Media*, advancing a method of his own for dealing with the

We easily find, from equations (x.), and (xv.), (xvi.), that

$$\rho(X-\ddot{u})+\kappa\ddot{u}+(\lambda+\mu)\frac{d\theta}{dx}+\mu\nabla^2 u-\frac{d^2}{dt^2}\left\{\overline{\lambda'+\mu'}\frac{d\theta}{dx}+\mu'\nabla^2 u\right\}$$

$$+\frac{\gamma}{4}\frac{d^2}{dt^2}\left(\frac{d\theta}{dx}-\nabla^2 u\right)=0 \quad \text{......(xxvi.)},$$

with similar equations for v and w.

Now take x for the direction of propagation of a plane-wave, and let its type be given by

$$u = A\cos\frac{2\pi}{l}(\Omega t-x),$$

$$v = B\cos\frac{2\pi}{l}(\Omega' t-x),$$

$$w = C\cos\frac{2\pi}{l}(\Omega' t-x).$$

Suppose also that there are no body forces, or $X=0$. We have

$$\frac{d\theta}{dx}=-\frac{4\pi^2}{l^2}u,\quad \frac{d\theta}{dy}=\frac{d\theta}{dz}=0,$$

$$\nabla^2 u=-\frac{4\pi^2}{l^2}u,\quad \nabla^2 v=-\frac{4\pi^2}{l^2}v,\quad \nabla^2 w=-\frac{4\pi^2}{l^2}w.$$

Substituting, we find, after division by $u\times\frac{4\pi^2}{l^2}$,

$$(\rho-\kappa)\Omega^2=(\lambda+2\mu)+(\lambda'+2\mu')\frac{4\pi^2}{l^2}\Omega^2-\frac{\gamma}{4}\left(\frac{4\pi^2}{l^2}\Omega^2-\frac{4\pi^2}{l^2}\Omega^2\right).$$

The last term disappears, and we have

$$\Omega^2=\frac{\lambda+2\mu}{\rho-\kappa-(\lambda'+2\mu')\frac{4\pi^2}{l^2}} \quad \text{..................(xxvii.)}.$$

The other body-shift equations give us

$$(\rho-\kappa)\Omega'^2=\mu+\mu'\frac{4\pi^2}{l^2}\Omega'^2+\frac{\gamma}{4}\frac{4\pi^2}{l^2}\Omega'^2,$$

---

influence of matter on the motion of the ether, but with references to the first paper. The equations he arrives at are almost identical with Boussinesq's, and he treats them in Boussinesq's manner. The method by which he reaches his equations seems to me difficult to follow, and open to many objections, especially the discussion of the mutual action terms on pp. 877–882. I think on this point Boussinesq's memoirs contain far less of assumption, and it is much easier to see the exact bearing of what there is.

or $$\Omega'^2 = \frac{\mu}{\rho - \kappa - \left(\mu' + \frac{\gamma}{4}\right)\frac{4\pi^2}{l^2}} \quad \text{..............(xxviii.).}$$

Thus the velocity of propagation of both longitudinal and transverse vibrations depends on the wave-length l, or dispersion takes place. In order that the velocity of propagation of transverse vibrations may increase with the length l of the wave, we must have $\mu' + \frac{\gamma}{4}$ negative.

While (xxviii.) thus gives an approximate value of the law of dispersion even in light waves, it is not of course anything like completely satisfactory. In order to fully explain dispersion, the formula for Ω' must also throw light on the nature of absorption and anomalous dispersion. Now these latter phenomena are closely allied to the relative motion of the *parts* of a molecule. Our expression, however, for the strain-energy has only taken into account the mean velocity of translation at any instant of all the molecules in a small element of the solid. It takes no account of internal molecular (atomic) motions. Hence we cannot expect any pure theory of elasticity to explain absorption or anomalous dispersion, or indeed to give a completely satisfactory formula for ordinary dispersion under all circumstances. We should expect, however, (xxviii.) to hold for a body of very simple molecular structure, when the period of the waves passing through it is not nearly equal to that of any of the internal molecular vibrations. In the atomic theory to which I have previously referred, the constants of (xxviii.) would be functions of the amplitudes of the internal molecular vibrations, and these would be seriously affected by a wave of period nearly equal to their own.* Thus μ', γ, and generally μ, would in a certain sense be a function of l, and the formula (xxviii.) would not fully represent the variation of Ω' with l.

11. *Double Refraction.* Let us suppose the medium to have three rectangular planes of elastic symmetry. Then we have, taking their intersections for axes—

$$\phi_1 = \tfrac{1}{2}\left\{as_x^2 + bs_y^2 + cs_z^2 + 2d's_ys_z + 2e's_zs_x + 2f's_xs_y + d_1\sigma_{yz}^2 + e_1\sigma_{zx}^2 + f\sigma_{xy}^2\right\}$$

$$\phi_2 = \tfrac{1}{2}\left\{\kappa_1\dot{u}^2 + \kappa_2\dot{v}^2 + \kappa_3\dot{w}^2\right\},$$

* I have considered the phenomena of dispersion and absorption when the law of cohesion is a function of internal molecular vibrations in a paper read to the London Mathematical Society in November, 1888, and printed Vol. xx., p. 40 *et seq.*

$$\phi_3 = \tfrac{1}{2}\left\{a_1\dot{s}_x^2 + b_1\dot{s}_y^2 + c_1\dot{s}_z^2 + 2d_1'\dot{s}_y\dot{s}_z + 2e_1'\dot{s}_z\dot{s}_x \right.$$

$$\left. + 2f_1'\dot{s}_x\dot{s}_y + d_1\dot{\sigma}_{yz}^2 + e_1\dot{\sigma}_{zx}^2 + f_1\dot{\sigma}_{xy}^2\right\},$$

$$\phi_4 = \tfrac{1}{2}\left(\gamma_1\dot{\tau}_{yz}^2 + \gamma_2\dot{\tau}_{zx} + \gamma_3\dot{\tau}_{xy}^2\right).$$

ϕ_{13} will not arise in our equations of motion if we put $a_{pq} = a_{qp}$ (see p. 308, e), which we will suppose to hold as in the case of isotropy. We find the following values for the stresses, where $D^2 = \dfrac{d^2}{dt^2}$,

$$\widehat{xx} = as_x + f's_y + e's_z - D^2\left(a_1s_x + f_1's_y + e_1's_z\right),$$

$$\widehat{yz} = d\sigma_{yz} - D^2\left(d_1\sigma_{yz}\right) + \frac{\gamma_1}{2}D^2\left(\tau_{yz}\right).$$

Hence the body stress-equations will be of the form,

$$(\rho-\kappa_1)\,\ddot{u} = \left(\overline{d+d'} - \overline{d_1+d_1'}D^2\right)\frac{ds_x}{dx} + \left(\overline{f+f'} - \overline{f_1+f_1'}D^2\right)\frac{ds_y}{dx}$$

$$+ \left(\overline{e+e'} - \overline{e_1+e_1'}D^2\right)\frac{ds_z}{dx}$$

$$+ \left(\overline{a-d-d'} - \overline{a_1-d_1-d_1'}D^2\right)\frac{d^2u}{dx^2} + \left(f - f_1D^2\right)\frac{d^2u}{dy^2}$$

$$+ \left(e - e_1D^2\right)\frac{d^2u}{dz^2} + \tfrac{1}{2}D^2\left[\gamma_2\frac{d}{dz}(\tau_{xz}) + \gamma_3\frac{d}{dy}(\tau_{xy})\right].$$

Or, substituting for τ_{xz} and τ_{xy}, this may be written in the form

$$(\rho-\kappa_1)\,D^2u = \left(a - a_1D^2\right)\frac{d^2u}{dx^2} + \left(f - \overline{f_1 + \tfrac{1}{4}\gamma_3}D^2\right)\frac{d^2u}{dy^2}$$

$$+ \left(e - \overline{e_1 + \tfrac{1}{4}\gamma_2}D^2\right)\frac{d^2u}{dz^2}$$

$$+ \left(\overline{f+f'} - \overline{f_1+f_1'-\tfrac{1}{4}\gamma_3}D^2\right)\frac{d^2v}{dx\,dy}$$

$$+ \left(\overline{e+e'} - \overline{e_1+e_1'-\tfrac{1}{4}\gamma_2}D^2\right)\frac{d^2w}{dx\,dz}.$$

Suppose the wave to be of period 2π [illegible] we may write the

above equation

$$(\rho-\kappa_1)\,D^2u = (a+a_1m^2)\frac{d^2u}{dx^2} + (f+\overline{f_1+\tfrac{1}{4}\gamma_3}\,m^2)\frac{d^2u}{dy^2}$$

$$+ (e+\overline{e_1+\tfrac{1}{4}\gamma_2}\,m^2)\frac{d^2u}{dz^2}$$

$$+ (f+f'+\overline{f_1+f_1'-\tfrac{1}{4}\gamma_3}\,m^2)\frac{d^2v}{dx\,dy}$$

$$+ (e+e'+\overline{e_1+e_1'+\tfrac{1}{4}\gamma_2}\,m^2)\frac{d^2w}{dx\,dz}\ldots\ldots(\text{xxiv.})$$

There will be two other equations of like type obtained by proper interchanges. We can further simplify the form of these equations by writing

$$f_1+\tfrac{1}{4}\gamma_3 = \mathfrak{f}_1,\quad e_1+\tfrac{1}{4}\gamma_2 = \mathfrak{e}_1,$$

$$f_1'-\tfrac{1}{2}\gamma_3 = \mathfrak{f}_1',\quad e_1'-\tfrac{1}{2}\gamma_2 = \mathfrak{e}_1'.$$

Let us further put

$$\mathfrak{A} = a+a_1m^2,$$

$$\mathfrak{F} = f+\mathfrak{f}_1m^2,\quad \mathfrak{F}' = f'+\mathfrak{f}_1'm^2,$$

$$\mathfrak{E} = e+\mathfrak{e}_1m^2,\quad \mathfrak{E}' = e'+\mathfrak{e}_1'm^2.$$

Thus we have

$$(\rho-\kappa_1)\,D^2u = \mathfrak{A}\frac{d^2u}{dx^2}+\mathfrak{F}\frac{d^2u}{dy^2}+\mathfrak{E}\frac{d^2u}{dz^2}$$

$$+(\mathfrak{F}+\mathfrak{F}')\frac{d^2v}{dx\,dy}+(\mathfrak{E}+\mathfrak{E}')\frac{d^2w}{dx\,dz}\ldots\ldots\ldots\ldots(\text{xxv.}).$$

There are two methods of reaching Fresnel's wave-surface from equations of this type.

(i.) Suppose the differences $\kappa_1, \kappa_2, \kappa_3$ to be very small, or zero.

Then Saint-Venant has shown* that if

$$\left.\begin{aligned} &(\mathfrak{B}-\mathfrak{D})(\mathfrak{C}-\mathfrak{D})-(\mathfrak{D}+\mathfrak{D}')^2 = 0\\ &(\mathfrak{C}-\mathfrak{E})\,(\mathfrak{A}-\mathfrak{E})-(\mathfrak{E}+\mathfrak{E}')^2 = 0\\ &(\mathfrak{A}-\mathfrak{F})\,(\mathfrak{B}-\mathfrak{F})-(\mathfrak{F}+\mathfrak{F}')^2 = 0\\ &(\mathfrak{A}-\mathfrak{E})(\mathfrak{B}-\mathfrak{F})(\mathfrak{C}-\mathfrak{D})+(\mathfrak{A}-\mathfrak{F})(\mathfrak{B}-\mathfrak{D})(\mathfrak{C}-\mathfrak{E})\\ &\qquad -2\,(\mathfrak{D}+\mathfrak{D}')(\mathfrak{E}+\mathfrak{E}')(\mathfrak{F}+\mathfrak{F}') = 0 \end{aligned}\right\}\ldots\ldots(\text{xxvi.}),$$

* "Mémoire sur la distribution des élasticités," *Journal de Liouville*, Vol. VIII., 1863, pp. 298–406. Or "History of Elasticity," Vol. II., Art. 148.

then equations of the type (xxv.) lead accurately to Fresnel's wave surface. These equations, however, give only very nearly transverse vibrations. They are practically identical with the conditions*

$$2\mathfrak{D}+\mathfrak{D}'=\sqrt{\mathfrak{B}\mathfrak{C}},\quad 2\mathfrak{E}+\mathfrak{E}'=\sqrt{\mathfrak{C}\mathfrak{A}},\quad 2\mathfrak{F}+\mathfrak{F}'=\sqrt{\mathfrak{A}\mathfrak{B}}\ldots\ldots \text{(xxvii.)}$$

It must be noted here, however, that our quantities are *not* Saint Venant's coefficients:

(*a*) In the first place, *our density has been changed* from ρ to $\rho-\kappa$.

(β) Our equations (xxvi.), or if we replace them by the ellipsoidal conditions (xxvii.), involve the wave-length. Our ellipsoidal conditions are of the type

$$2d+d'+(2d_1+d_1')\,m^2=\sqrt{(b+b_1m^2)(c+c_1m^2)}.$$

It is only by neglecting the after-strain terms, which of course will be small as compared with the others, that we obtain Saint-Venant's

$$2d+d'=\sqrt{bc}.$$

It is to be noted also that $2d_1+d_1'$ is a pure after-strain coefficient, for the twist-speed coefficient disappears from it.

(ii.) We may suppose that the vibrating medium remains to all intents and purposes isotropic, but that the κ's are sensible and different. In this case our equations become

$$(\rho-\kappa_1)\,D^2u=\left\{\lambda+\mu-(\lambda'+\mu')\,D^2\right\}\frac{d\theta}{dx}+(\mu-\mu'D^2)\,\nabla^2u,$$

$$(\rho-\kappa_2)\,D^2v=\left\{\lambda+\mu-(\lambda'+\mu')\,D^2\right\}\frac{d\theta}{dy}+(\mu-\mu'D^2)\,\nabla^2v,$$

$$(\rho-\kappa_3)\,D^2w=\left\{\lambda+\mu-(\lambda'+\mu')\,D^2\right\}\frac{d\theta}{dz}+(\mu-\mu'D^2)\,\nabla^2w.$$

Suppose the vibration of period $\frac{2\pi}{m}$, then $D^2\left(=\frac{d^2}{dt^2}\right)$ may be replaced by $-m^2$, and let

$$F\,(\rho-\kappa_1)=G\,(\rho-\kappa_2)=H\,(\rho-\kappa_3)=\lambda+\mu+(\lambda'+\mu')\,m^2,$$

$$F_1\,(\rho-\kappa_1)=G_1\,(\rho-\kappa_2)=H_1\,(\rho-\kappa_3)=\mu+\mu'm^2.$$

Then we have

$$\left.\begin{aligned}\frac{d^2u}{dt^2}&=F\frac{d\theta}{dx}+F_1\nabla^2u\\ \frac{d^2v}{dt^2}&=G\frac{d\theta}{dy}+G_1\nabla^2v\\ \frac{d^2w}{dt^2}&=H\frac{d\theta}{dz}+H_1\nabla^2w\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxviii.)}$$

* "History of Elasticity," Vol. II., Art. 149, or the "Mémoire," pp. 406 to 411.

These are of the form given by Sarrau (*Journa l de Liouville*, Vol. XIII., 1868, p. 78). They lead to Fresnel's wave-surface, if we take

$$F+F_1 = G+G_1 = H+H_1 = 0,$$

or

$$(\lambda+2\mu)+m^2(\lambda'+2\mu') = 0.$$

This frees us also from the "pressural" wave. But such a condition is unlikely to be fulfilled, since λ' and μ' can hardly be negative, and, if they were, it would only be fulfilled for one wave-length. Our results, however, differ from Sarrau's by the fact that, although we reach the equation for the wave-velocity, Ω ($2\pi/m = l/\Omega$ if l be the wave-length), namely,

$$\frac{L^2}{\Omega^2-F}+\frac{M^2}{\Omega^2-G}+\frac{N^2}{\Omega^2-H} = 0, \text{ (see his p. 91,)}$$

where L, M, N are the direction-cosines of the wave-front, still our F, G, H are functions of m or of Ω/l; so that, even if we do assume the disappearance of the pressural wave, we still do not accurately get the wave-surface of Fresnel, unless we neglect the after-strain terms. We should expect slight variations in the wave-surface for different wave-lengths.

If we do not suppose the conditions

$$F+F' = G+G' = H+H' = 0$$

to hold, we still get very approximately Fresnel's wave-surface, and the vibrations very approximately transverse. This has been shown by Boussinesq: see his "Mémoire sur les ondes dans les milieux isotropes déformés" (*Journal de Liouville*, Vol. XIII., 1868, pp. 221 and 229). In this memoir, however, he reaches equations (xxviii.) by very different considerations. In another memoir in the same volume of *Liouville* (the well-known "Théorie nouvelle des ondes lumineuses"), he reaches by another hypothesis equations akin to our (xxviii.): see his p. 330. He speaks of his coefficients K and L, corresponding to our

$$\lambda+\mu+\lambda'+\mu'm^2 \quad \text{and} \quad \mu+\mu'm^2,$$

as containing the length of the wave; but it seems to me they also contain the *velocity of the wave* like those above, and therefore, even for a given wave-length, the velocity will not be given accurately by Fresnel's wave-surface—were we to disregard the pressural wave. Thus it appears that Boussinesq's equations do not give so accurately the wave-surface as might be concluded from his remarks on p. 330

of the latter memoir, and therefore differ from Lord Rayleigh's form of the equations: see Glazebrook's "Report on Optical Theories," British Association 1885, pp. 178 and 215.

The idea of keeping the medium to all intents and purposes isotropic was, I believe, due originally to Rankine, but the first satisfactory treatment appears to be that of Boussinesq (*Liouville*, Vol. XIII., p. 328).

12. *Reflection and Refraction.*—Case (i.): Vibrations in the plane of incidence.

Let the axis of x be taken perpendicular to the plane surface which separates the two media, and the axis of z parallel to the wave fronts. Let us put

$$\mu''' = \mu' + \gamma/4,$$
$$\lambda''' = \lambda' - \gamma/2.$$

Then for an isotropic medium, if $D = \frac{d}{dt}$, we have the equations

$$\left.\begin{aligned}(\rho-\kappa)D^2 u &= \{\lambda+2\mu-(\lambda'''+2\mu''')D^2\}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}\right)\\ &\quad+(\mu-\mu'''D^2)\frac{d}{dy}\left(\frac{du}{dy}-\frac{dv}{dx}\right)\\ (\rho-\kappa)D^2 v &= \{\lambda+2\mu-(\lambda'''+2\mu''')D^2\}\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}\right)\\ &\quad+(\mu-\mu'''D^2)\frac{d}{dx}\left(\frac{dv}{dx}-\frac{du}{dy}\right)\end{aligned}\right\}\ldots(\text{xxix.}).$$

These shall represent the equations for the shifts u, v, in the first medium; the constants of those for u_1, v_1 of the second medium will be distinguished by subscripts. For continuity at the junction of the two media, we must have

$$\left.\begin{aligned}&u = u_1, \quad v = v_1,\\ &(\lambda+2\mu-\overline{\lambda'+2\mu'}D^2)\frac{du}{dx}+(\lambda-\lambda'D^2)\frac{dv}{dy}\\ &\qquad=(\lambda_1+2\mu_1-\overline{\lambda_1'+2\mu_1'}D^2)\frac{du_1}{dx}+(\lambda_1-\lambda_1'D^2)\frac{dv_1}{dy}\\ &(\mu-\mu'D^2)\left(\frac{du}{dy}+\frac{dv}{dx}\right)-\frac{\gamma D^2}{4}\left(\frac{dv}{dx}-\frac{du}{dy}\right)\\ &\qquad=(\mu_1-\mu_1'D^2)\left(\frac{du_1}{dy}+\frac{dv_1}{dx}\right)-\frac{\gamma_1 D^2}{4}\left(\frac{dv_1}{dx}-\frac{du_1}{dy}\right)\end{aligned}\right\}\ldots\ldots(\text{xxx.}).$$

Assuming, with Green* and Lord Rayleigh,†

$$u = \frac{d\phi}{dx} + \frac{d\psi}{dy}, \quad v = \frac{d\phi}{dy} - \frac{d\psi}{dx},$$

we find

$$\left.\begin{aligned}(\rho-\kappa)\, D^2\phi &= \left\{\lambda+2\mu-\overline{\lambda'''+2\mu'''}\, D^2\right\}\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}\right)\\ (\rho-\kappa)\, D^2\psi &= \left\{\mu-\mu'''D^2\right\}\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}\right)\end{aligned}\right\}\ldots\text{(xxxi.)}.$$

Take

$$\psi = Ae^{i(ax+by+ct)} + Be^{i(-ax+by+ct)},$$

$$\phi = A'e^{i(a'x+by+ct)},$$

for the first medium; and

$$\psi_1 = Ce^{i(a_1x+by+ct)},$$

$$\phi_1 = C'e^{i(a_1'x+by+ct)}$$

for the second medium.

We have on substitution

$$\left.\begin{aligned}(\rho-\kappa)\, c^2 &= \left\{\lambda+2\mu+c^2(\lambda'''+2\mu''')\right\}(a'^2+b^2) = (\mu+c^2\mu''')\,(a^2+b^2)\\ (\rho_1-\kappa_1)\, c^2 &= \left\{\lambda_1+2\mu_1+c^2(\lambda_1'''+2\mu_1''')\right\}(a_1'^2+b^2) = (\mu_1+c^2\mu_1''')\,(a_1^2+b^2)\end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\text{(xxxii.)}.$$

Let
$$\omega^2 = \frac{\lambda+2\mu}{\rho-\kappa}, \quad \beta = \frac{\lambda'''+2\mu'''}{\lambda+2\mu} = \frac{\lambda'+2\mu'}{\lambda+2\mu},$$

since $\lambda'''+2\mu'''$ does not contain γ; further, let

$$\Omega^2 = \frac{\mu}{\rho-\kappa}, \quad \beta' = \frac{\mu'''}{\mu}.$$

Then

$$c^2 = \omega^2\,(1+\beta c^2)(a'^2+b^2) = \Omega^2\,(1+\beta' c^2)(a^2+b^2)$$

$$= \omega_1^2\,(1+\beta_1 c^2)(a_1'^2+b^2) = \Omega_1^2\,(1+\beta_1' c^2)(a_1^2+b^2)\ldots\ldots\text{(xxxiii.)}.$$

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* *Collected Papers*, p. 261.
† *Phil. Mag.*, 1871, Vol. XLII., p. 88.

Let $b/a = \tan\theta$, $b/a_1 = \tan\theta_1$, we find

$$\frac{\sin\theta_1}{\sin\theta} = \sqrt{\frac{a^2+b^2}{a_1^2+b^2}} = \frac{\Omega_1}{\Omega}\sqrt{\frac{1+\beta_1' c^2}{1+\beta' c^2}} = \frac{1}{r},$$

where r is the refractive index. Let $r_0 = \Omega/\Omega_1$, then

$$r = r_0\sqrt{\frac{1+\beta' c^2}{1+\beta_1' c^2}} \quad\text{......................} \quad \text{(xxxiv.)}$$

Thus, since c is a function of the period of the vibration, r will als be, and there will be dispersion unless $\beta' = \beta_1'$ for the two media Now Green, Lord Rayleigh, and Boussinesq take the elastic constant λ and μ the same for the two media, and we might, following them possibly put the after-strain coefficients λ' and μ' the same, but i seems unlikely that γ and γ_1 will be equal. For we have suppose different densities in the two media $\rho-\kappa$ and $\rho_1-\kappa_1$, and so most pro bably κ and κ_1 different. But, if the coefficients of translation-speed are different, it is extremely likely that the rotatory coefficients γ, γ will be different, for these coefficients will express the work done du to a mere rotation of the medium as a rigid body, as well as the wor due to twist-speeds. Thus, while we shall assume λ and μ equal t λ_1 and μ_1, we shall treat γ and γ_1 as differing. Thus there will b dispersion.

Now let us suppose w and w_1 very large, then we must have a'^2+b and $a_1'^2+b^2$ almost zero, or to a second approximation

$$a' = +\sqrt{-1}\,b\left\{1-\frac{1}{2b^2}\,\frac{c^2}{\omega^2(1+\beta c^2)}\right\},$$

$$a_1' = -\sqrt{-1}\,b\left\{1-\frac{1}{2b^2}\,\frac{c^2}{\omega_1^2(1+\beta_1 c^2)}\right\}.$$

Now let us write, with Lord Rayleigh (*loc. cit.*, p. 89).

$$X = A+B, \quad Y = A-B.$$

The surface equivalence of the shifts gives

$$\left.\begin{aligned} a'A'+bX &= a_1'C'+bC \\ bA'-aY &= bC'-a_1C \end{aligned}\right\} \quad\text{..................} \quad \text{(xxxv.)}$$

while, from the surface equivalence of the stresses (xxx.), we have

$$\left.\begin{aligned}&\{\lambda+2\mu+c^2(\lambda'+2\mu')\}\frac{d^2\phi}{dx^2}+2(\mu+c^2\mu')\frac{d^2\psi}{dx\,dy}+(\lambda+\lambda'c^2)\frac{d^2\phi}{dy^2}\\ &=\{\lambda_1+2\mu_1+c^2(\lambda_1'+2\mu_1')\}\frac{d^2\phi_1}{dx^2}+2(\mu_1+c^2\mu_1')\frac{d^2\psi_1}{dx\,dy}+(\lambda_1+\lambda_1'c^2)\frac{d^2\phi_1}{dy^2}\\ &(\mu+\mu'c^2)\left\{2\frac{d^2\phi}{dx\,dy}+\frac{d^2\psi}{dy^2}-\frac{d^2\psi}{dx^2}\right\}-\frac{\gamma c^2}{4}\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}\right)\\ &=(\mu_1+\mu_1'c^2)\left\{2\frac{d^2\phi_1}{dx\,dy}+\frac{d^2\psi_1}{dy^2}-\frac{d^2\psi_1}{dx^2}\right\}-\frac{\gamma_1c^2}{4}\left(\frac{d^2\psi_1}{dx^2}+\frac{d^2\psi_1}{dy^2}\right)\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxxvi.)}.$$

Let us first deal with the second of these equations and substitute the values of ψ and ϕ; we find

$$(\mu+\mu'c^2)\{2a'bA'+(b^2-a^2)X\}-\frac{\gamma c^2}{4}X(a^2+b^2)$$
$$=(\mu_1+\mu_1'c^2)\{2a_1'bC'+(b^2-a_1^2)C\}-\frac{\gamma_1c^2}{4}C(a_1^2+b^2).$$

Putting the after-strain and elastic coefficients λ', μ' and λ, μ the same in the two media, we have

$$2a'bA'+(b^2-a^2)X-\frac{\gamma c^2}{4(\mu+\mu'c^2)}X(a^2+b^2)$$
$$=2a_1'bC'+(b^2-a_1^2)C-\frac{\gamma_1c^2}{4(\mu+\mu'c^2)}C(a_1^2+b^2).$$

But $a'A'-a_1'C'=b(C-X)$, by (xxxv.). Hence

$$X(a^2+b^2)\left\{1+\frac{\gamma c^2}{4(\mu+\mu'c^2)}\right\}=C(a_1^2+b^2)\left\{1+\frac{\gamma_1c^2}{4(\mu+\mu'c^2)}\right\}.$$

Or, $$X=Cr^2\times\left\{\frac{1+\beta_1'c^2}{1+\beta'c^2}\right\}=Cr_0^2\ldots\ldots\ldots\ldots\text{(xxxvii.)}.$$

Returning now to the first surface stress-equation of (xxxvi.), we have on substitution

$$\{\lambda+2\mu+c^2(\lambda'+2\mu')\}(a'^2+b^2)A'-\{\lambda_1+2\mu_1+c^2(\lambda_1'+2\mu_1')\}(a_1'^2+b^2)C'$$
$$=-2(\mu+c^2\mu')b(aY-bA')+2(\mu_1+c^2\mu_1')b(a_1C-bC').$$

The right-hand side of this equation vanishes if we put μ and μ' equal to μ_1 and μ_1' respectively by (xxxv.). Hence, remembering

(xxxii.) we have

$$(\rho-\kappa) A' = (\rho_1-\kappa_1) C',$$

or

$$A'/C' = r_0^2 \text{ ....(xxxviii.).}$$

This equation agrees with Lord Rayleigh (*loc. cit.*, p. 90). From (xxxv.) we easily deduce

$$Y = C\left\{\frac{a_1}{a} + \sqrt{-1}\,\frac{(r_0^2-1)^2}{r_0^2+1}\,\frac{b}{a}\right\} = C\left\{\frac{\cot\theta_1}{\cot\theta} + \sqrt{-1}\tan\theta\,\frac{(r_0^2-1)^2}{r_0^2+1}\right\}.$$

Joining this with $X = Cr_0^2$, we find, if $i = \sqrt{-1}$,

$$\left.\begin{aligned} A &= \frac{C}{2}\left\{r_0^2 + \frac{\cot\theta_1}{\cot\theta} + i\tan\theta\,\frac{(r_0^2-1)^2}{r_0^2+1}\right\} \\ B &= \frac{C}{2}\left\{r_0^2 - \frac{\cot\theta_1}{\cot\theta} - i\tan\theta\,\frac{(r_0^2-1)^2}{r_0^2+1}\right\}\end{aligned}\right\}\text{.........(xxxix.),}$$

where

$$r = \frac{\sin\theta}{\sin\theta_1}, \quad\text{and}\quad r = r_0\sqrt{\frac{1+\beta' c^2}{1+\beta_1' c^2}}.$$

These equations differ from those given by Lord Rayleigh (p. 91), unless we make $\beta_1' = \beta'$, which is equivalent to putting γ and γ_1 equal or zero. If we do this, then the equations are subject to the same disadvantage as those of Green, namely the coefficient of $i\tan\theta$ is given by $\frac{(r^2-1)^2}{r^2+1}$, and this does not agree with experiment. On the other hand, if the difference in β', β_1' be real, then the coefficient of $i\tan\theta$ can be considered as depending upon a refractive index r_0 differing from r; but then the first term in the brackets of the values of A and B is incorrect. Throwing A and B into Lord Rayleigh's form, we have

$$A = \frac{C}{2}\left\{r^2 + \frac{\cot\theta^2}{\cot\theta} - (r^2-r_0^2) + i\tan\theta\,\frac{(r_0^2-1)^2}{r_0^2+1}\right\}$$

$$= \frac{C}{2}\tan\theta\,(r^2-1)\left\{\cot(\theta-\theta_1) - \frac{r-r_0^2}{r-1}\cot\theta + i\,\frac{(r_0^2-1)^2}{(r_0^2+1)(r^2-1)}\right\},$$

$$B = \frac{C}{2}\tan\theta\,(r^2-1)\left\{\cot(\theta+\theta_1) - \frac{r^2-r_0^2}{r^2-1}\cot\theta - i\,\frac{(r_0^2-1)^2}{(r_0^2+1)(r^2-1)}\right\}.$$

Put

$$\delta = \frac{r^2-r_0^2}{r^2-1}, \quad\text{and}\quad p_0 = \frac{(r_0^2-1)^2}{r_0^2+1},$$

and let e and e' be the changes of phase; then

$$\frac{B^2}{A^2}=\frac{\cot^2(\theta+\theta_1)+\delta^2\cot^2\theta-2\delta\cot\theta\cot(\theta+\theta_1)+p_0^2(r^2-1)^{-2}}{\cot^2(\theta-\theta_1)+\delta^2\cot^2\theta-2\delta\cot\theta\cot(\theta-\theta_1)+p_0^2(r^2-1)^{-2}}\ldots(\text{xl.}).$$

Further,
$$\left.\begin{aligned}\cot e &= \frac{r^2-1}{p_0}\left\{\cot(\theta-\theta_1)-\delta\cot\theta\right\}\\ \cot e' &= -\frac{r^2-1}{p_0}\left\{\cot(\theta+\theta_1)-\delta\cot\theta\right\}\end{aligned}\right\}\ldots\ldots\ldots\ldots(\text{xli.}).$$

Now, r_0 is the value of r for a wave of vibrations of infinitely long period, if we suppose β' to have a sensible value; hence it is not the refractive index of any true wave, and r will be generally considerably greater than r_0. At the same time r will be a function of the period of the vibration $2\pi/c$, unless we assume β' to vary with c, so that $(1+\beta'c^2)/(1+\beta_1'c^2)$ is nearly constant, or $1/c^2$ be very small as compared with β' and β_1', in which case this expression takes the value β'/β_1'. This of course is equivalent to supposing the twist terms, or the after-strain terms, or both, much more important than the elastic terms in the original equation.

If we disregard the terms in δ, and suppose $p_0^2(r^2-1)^{-2}$ equals Lord Rayleigh's M^2, we find

$$M(r^2-1)=p_0.$$

Now, Lord Rayleigh puts $M=\dfrac{\mu_0^2-1}{\mu_0^2+1}$, where μ_0 is a certain constant. For sulphuret of arsenic $\mu=2{\cdot}454$ and $\mu_0=1{\cdot}083$, after a result of Jamin's quoted by Lord Rayleigh. Hence we find

$$\cdot 3996=\frac{(r_0^2-1)^2}{r_0^2+1}\quad\text{or}\quad r_0^2=2{\cdot}1\ \text{about.}$$

Thus
$$\frac{1+\beta'c^2}{1+\beta_1'c^2}=\frac{r^2}{r_0^2}=1{\cdot}17\ \text{about.}$$

Case (i.) $1/c^2$ very small: $\beta'/\beta_1'=1{\cdot}17$, or β_1' would be less for reflecting medium.

Case (ii.) $\beta'=0$, $\beta_1'c^2=-\cdot 145$ about. If μ' be small as compared with $\gamma/4$, we should have

$$\gamma c^2=-\cdot 58\mu.$$

This largeness of γ and its negative character (*i.e.*, to correspond to negative κ's) do not seem to me improbable.

The above value of M is, however, calculated from Lord Rayleigh's

results on the hypothesis that $\delta = 0$, and in most cases it seems probable that the terms depending on δ would be of importance. The appearance of δ in the formulæ is not satisfactory; but, putting $\beta' = \beta_1'$ we obtain at least as good results as from the old theory of elastic solids.

13. Case (ii.). Taking the case of vibrations perpendicular to the plane of incidence, we have, with the previous notation,

$$\left.\begin{aligned}(\rho-\kappa)\,D^2 w &= (\mu-\mu'''D^2)\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}\right)\\(\rho_1-\kappa_1)\,D^2 w_1 &= (\mu_1-\mu_1'''D^2)\left(\frac{d^2w_1}{dx^2}+\frac{d^2w_1}{dy^2}\right)\end{aligned}\right\}\quad\text{.........(xlii.)}$$

The surface equality of stress and shift give, for $x = 0$,

$$\left.\begin{aligned}(\mu-\mu'''D^2)\frac{dw}{dx} &= (\mu_1-\mu_1'''D^2)\frac{dw_1}{dx}\\ w &= w_1\end{aligned}\right\}\quad\text{............(xliii.)}$$

Assuming $$w = Ae^{i(ax+by+ct)}+Be^{i(-ax+by+ct)},$$
$$w_1 = Ce^{i(a_1x+by+ct)},$$

we find $$\left.\begin{aligned}(\rho-\kappa)\,c^2 &= (\mu+c^2\mu''')(a^2+b^2)\\(\rho_1-\kappa_1)\,c^2 &= (\mu_1+c^2\mu_1''')(a_1^2+b^2)\end{aligned}\right\}\quad\text{............(xliv.)}$$

Eliminating c^2, we find, assuming $\mu = \mu_1$,

$$\frac{a_1^2+b^2}{a^2+b^2} = \frac{\rho_1-\kappa_1}{\rho-\kappa}\,\frac{1}{1+\dfrac{\mu_1'''-\mu'''}{\rho-\kappa}(a^2+b^2)}\text{ nearly.}$$

If l be the length of the incident wave, this gives

$$r^2 = r_0^2\times\frac{1}{1+\dfrac{\mu_1'''-\mu'''}{\rho-\kappa}\,\dfrac{4\pi^2}{l^2}}\quad\text{....................(xlv.)}$$

Applying conditions (xliii.), we find

$$(\mu+\mu'''c^2)\,a\,(A-B) = (\mu_1+\mu_1'''c^2)\,a_1C,$$
$$A+B = C,$$

or $$\frac{A+B}{A-B} = \frac{\mu+\mu'''c^2}{\mu_1+\mu_1'''c^2}\,\frac{a}{a_1} = \frac{r^2}{r_1^2}\,\frac{a}{a_1} = \frac{r^2}{r_0^2}\,\frac{\cot\theta}{\cot\theta_1}.$$

Hence
$$\frac{B}{A}=\frac{\dfrac{\tan\theta_1}{\tan\theta}-\dfrac{r_0^2}{r^2}}{\dfrac{\tan\theta_1}{\tan\theta}+\dfrac{r_0^2}{r^2}}=\frac{\dfrac{\tan\theta_1}{\tan\theta}-1-\dfrac{\mu_1'''-\mu'''}{\rho-\kappa}\dfrac{4\pi^2}{l^2}}{\dfrac{\tan\theta_1}{\tan\theta}+1+\dfrac{\mu_1'''-\mu'''}{\rho-\kappa}\dfrac{4\pi^2}{l^2}},\quad\text{.........(xlvi.).}$$

Now, clearly, this result will not agree with the ordinary sine formula of Fresnel, unless we put

$$\mu_1'''=\mu''',\ \text{or unless}\ \frac{\mu_1'''-\mu'''}{\rho-\kappa}$$

be so small as to be negligible; but this requires us to put $r=r_0$, or at least r and r_0, to be very nearly equal. But to assume this destroys any advantage we might gain from r_0 not being equal to r in the previous case of vibrations in the plane of incidence.*

Thus the generalised equations of elasticity appear only to introduce dispersive terms into the formulæ, but do not otherwise improve on the old theory of reflection.†

14. I now propose to find the alterations which it will be needful to make in our equations when terms involving the products of the shift-speeds and twist-speeds are included. For a "dissymmetric" isotropic solid these must be of the form

$$\begin{aligned}\phi_{24}=\ &\dot u\,\{q_1\dot\tau_{yz}+q_2(\dot\tau_{xy}+\dot\tau_{xz})\}\\&+\dot v\,\{q_1\dot\tau_{zx}+q_2(\dot\tau_{yz}+\dot\tau_{yx})\}\\&+\dot w\,\{q_1\dot\tau_{xy}+q_2(\dot\tau_{zx}+\dot\tau_{zy})\}.\end{aligned}$$

Now suppose the following displacements:

$$u=0,\quad v=Vt+az,\quad w=Wt-ay;$$

---

* Can any distinction be drawn here between the refractive indices for rays polarised in and perpendicular to the plane of incidence?

† Sir William Thomson has shown, in the *Phil. Mag.* for November, 1888, (and his results have been extended by Mr. Glazebrook after the manner of Boussinesq in the *Phil. Mag.* for December, 1888,) that Fresnel's tangent formula is reached if we put $\lambda+2\mu=0$. I have shown in a "Note on Twists," *Math. Messenger*, XIII., 1884, p. 85, that the internal energy of an infinite elastic solid in a state of zero strain at an infinite distance

$$=\tfrac{1}{2}\int\{(\lambda+2\mu)\,\theta^2+\mu\tau^2\}\,d\,(\text{vol.}).$$

Hence the *total* energy would still be positive if $\lambda+2\mu=0$, or λ were negative. Sir William Thomson supposes the ether rigidly fixed at infinity, and so is able to consider its motion as stable. There would thus be no difficulty left in explaining reflection, were we satisfied with the tangent formula, or did we comprehend how the ether can be considered as rigidly fixed at infinity. A like rigidity of the ether would also have to hold over all opaque surfaces with which it might be in contact, a somewhat difficult conception.

then $$\phi_{34} = q_2 \dot{r}_{yz}(V-W) + q_2 \dot{a}^2 (y+z)$$
$$= q_2 \dot{a}(W-V), \text{ for points on the axis of } x.$$

This is the work corresponding to a spin $\dot{a}$ round the axis of x and a velocity perpendicular to it with components W and V. If $W=V$, this work $=0$; if $W=-V$, then the work is finite: it is hard to understand how this complete difference could exist even in rotating liquids. Hence we shall assume $q_2=0$. This gives us

$$\phi_{34} = q_1 (\dot{u}\dot{r}_{yz} + \dot{v}\dot{r}_{zx} + \dot{w}\dot{r}_{xy}).$$

There is no alteration in this expression if the sign of all the shifts be reversed, but there is alteration if the sign of only one of them be reversed. If $\dot{T}$ be the resultant twist and $\dot{U}$ the resultant shift velocity, this is of the form

$$\phi_{34} = q_1 \dot{U}\dot{T} \cos\phi \text{ ....} \text{(xlvii.)},$$

where ϕ is the angle between the resultant twist-velocity and the resultant shift-velocity. We have, in the earlier part of our work, rejected such a term for a *perfectly isotropic* solid, as it would change its sign if we changed the direction of $\dot{U}$. Now there seems absolutely no mechanical reason why an element of a perfectly isotropic body should be more affected by a right-handed than a left-handed system of twists applied to its elements. Yet this is practically the assumption made by all theories which endeavour to explain rotation of the plane of polarisation in the case of substances assumed to be "perfectly isotropic." This difficulty is got over by Boussinesq by the introduction of media which he terms *isotropes-dissymétriques* (*Liouville*, Vol. XIII., p. 319). Such media are isotropic if we rotate the system of axes as a whole, but not if we change the sense of one without those of the others. Boussinesq supposes liquids which rotate the plane of polarisation to be *isotropically dissymmetric*. The reason for this peculiarity he does not state, and the same want of mechanical explanation would apply to Von Lang's theory.* We shall, however, assume that such a term as ϕ_{34} can occur, and examine whither it leads us.

15. *Rotation of Plane of Polarisation.*

In the full equations of an isotropic elastic medium, we have now to

* See Glazebrook's criticisms, "Report on Optical Theories," pp. 177 and 216.

add the terms arising from ϕ_{24}. Referring to equation (vii.), we see that we must include

$$\frac{d}{dt}\left(\frac{d\phi_{24}}{d\dot{u}}\right)+\frac{1}{2}\,\frac{d}{dt}\left(\frac{d}{dy}\,\frac{d\phi_{24}}{d\dot{\tau}_{xy}}-\frac{d}{dz}\,\frac{d\phi_{24}}{d\dot{\tau}_{zx}}\right).$$

But this

$$= q_1\,\frac{d^2}{dt^2}(\tau_{yz})+\frac{q_1}{2}\,\frac{d}{dt}\left(\frac{d\dot{w}}{dy}-\frac{d\dot{v}}{dz}\right)$$

$$= q_1\,\frac{d^2}{dt^2}(\tau_{yz})+q_1\,\frac{d^2}{dt^2}(\tau_{yz}) = 2q_1\,\frac{d^2}{dt^2}(\tau_{yz}).$$

Thus the type of body shift equation is

$$(\rho-\kappa)\,D^2u = \left\{\overline{\lambda+\mu}-(\lambda'+\mu'-\gamma/4)\,D^2\right\}\frac{d\theta}{dx}+(\mu-\overline{\mu'+\gamma/4}\,D^2)\,\nabla^2 u$$

$$+q\,\frac{d^2}{dt^2}\left(\frac{dw}{dy}-\frac{dw}{dz}\right) \quad\text{.....................(xlviii.).}$$

Assuming the period of the wave to be $2\pi/c$, this equation takes the form

$$D^2u = p_1\,\frac{d\theta}{dx}+p_2\nabla^2u-p_3\left(\frac{dw}{dy}-\frac{dv}{dz}\right),$$

where

$$\left.\begin{aligned} p_1 &= \frac{\lambda+\mu+(\lambda'+\mu'-\gamma/4)\,c^2}{\rho-\kappa} \\ p_2 &= \frac{\mu+(\mu'+\gamma/4)\,c^2}{\rho-\kappa} \\ p_3 &= \frac{qc^2}{\rho-\kappa} \end{aligned}\right\} \quad\text{...............(xlix.).}$$

Now suppose the plane of yz parallel to that of the waves, and let

$$u = Ae^{i(ax+ct)},$$

$$v = Be^{i(ax+ct)},$$

$$w = Ce^{i(ax+ct)}.$$

Substituting, we have

$$-c^2A = -p_1a^2A-p_2a^2A,$$

or

$$\{c^2-(p_1+p_2)\,a^2\}\,A = 0,$$

$$-c^2B = -p_2a^2B+p_3iaC,$$

$$-c^2C = -p_2a^2C-p_3iaB.$$

The first equation is satisfied either by the longitudinal wave given by

$$c^2 = (p_1+p_2)\,a^2,$$

or by putting $A = 0$.

Assuming this latter to be true, as we wish to deal only with waves of transverse vibrations, we have

$$(c^2-p_2a^2)^2 = p_3^2a^2,$$

or

$$c^2-p_2a^2 = \pm p_3a,$$

$$p_2a^2 \pm p_3a - c^2 = 0,$$

$$a = \frac{\pm p_3 + \sqrt{p_3^2+4c^2p_2}}{2p_2}.$$

We need only take the + sign of the radical, if we are treating of a wave in one direction only. We have thus two values of a, a_1 and a_2, and two of B and C, $-iB_1 = C_1$ and $iB_2 = +C_2$. Thus we have

$$v = B_1e^{i(a_1x+ct)} + B_2e^{i(a_2x+ct)},$$

$$w = -iB_1e^{i(a_1x+ct)} + iB_2e^{i(a_2x+ct)},$$

where

$$a_1 = \frac{p_3 + \sqrt{p_3^2+4c^2p_2}}{2p_2},$$

$$a_2 = \frac{-p_3 + \sqrt{p_3^2+4c^2p_2}}{2p_2}.$$

Hence, taking $B_1 = Ce^{i\alpha_1}$, $B_2 = Ce^{i\alpha_2}$, we find, by equating real parts,

$$v = C\left\{\cos(a_1x+ct+\alpha_1) + \cos(a_2x+ct+\alpha_2)\right\},$$

$$w = C\left\{\sin(a_1x+ct+\alpha_1) - \sin(a_2x+ct+\alpha_2)\right\}.$$

These correspond to two circularly polarised rays of opposite phase. Combining the trigonometrical terms, we have

$$v = 2C\cos\left(\frac{a_1+a_2}{2}x + ct + \frac{\alpha_1+\alpha_2}{2}\right)\cos\left(\frac{a_1-a_2}{2}x + \frac{\alpha_1-\alpha_2}{2}\right),$$

$$w = 2C\cos\left(\frac{a_1+a_2}{2}x + ct + \frac{\alpha_1+\alpha_2}{2}\right)\sin\left(\frac{a_1-a_2}{2}x + \frac{\alpha_1-\alpha_2}{2}\right).$$

Thus the velocity of the wave is

$$\left.\begin{array}{ll} & \dfrac{2c}{a_1+a_2},\ \text{or}\ \dfrac{2p_2c}{\sqrt{p_3^2+4c^2p_2}} \\ \text{and its rotation is} & \dfrac{a_1-a_2}{2}x,\ \text{or}\ \dfrac{p_3x}{2p_2} \end{array}\right\} \quad \text{......................(l.).}$$

These give for the velocity

$$\frac{\mu+(\mu'+\gamma/4)\,c^2}{\sqrt{(\rho-\kappa)\,\{\mu+(\mu'+\gamma/4)\,c^2\}+\dfrac{q^2c^2}{4}}}=\sqrt{\frac{\mu+(\mu'+\gamma/4)\,c^2}{\rho-\kappa}}\text{......(li.)},$$

or it equals the velocity of waves of transverse vibrations if q^2c^2 be small. While the rotation

$$\left.\begin{array}{l} =\dfrac{qc^2}{2\,\{\mu+(\mu'+\gamma/4)\,c^2\}}\times(\text{thickness}) \\ =\dfrac{q}{2}\,\dfrac{4\pi^2}{(\rho-\kappa)\,l^2}\times\text{thickness} \end{array}\right\} \quad \text{............(lii.).}$$

These results are in complete agreement with those of Boussinesq's first method (*loc. cit.*, p. 324). His second method does not seem applicable to the generalised equations of elasticity, unless we suppose terms consisting of products of strains and twist-speeds to occur in the strain-energy. At the same time, it must be remembered that (lii.) is only an approximation arising from the neglect of the q^2 term in (li.). If we do not neglect this term, the velocity in the medium will not be that of a wave of transverse vibrations outside the medium, and the rotation will then have to be expanded in powers of c^2 or inverse powers of l^2, the wave-length squared.

16. *Theory of Quartz.*

Suppose the term ϕ_{24} to occur in a crystal, then, if it has three axes, we may treat it as of the form

$$q_1\ddot{u}\tau_{yz}+q_2\ddot{v}\tau_{zx}+q_3\ddot{w}\tau_{xy}.$$

Further, if we suppose with Boussinesq the effect on the medium to be solely represented by change of density, we have Sarrau's equa-

tions given as (xxviii.) above, with additional terms due to twist,

$$D^2u = F\frac{d\theta}{dx} + F_1\nabla^2 u + \frac{q_1}{2(\rho-\kappa_1)}\left(\frac{dw}{dy} - \frac{dv}{dz}\right)$$

$$+ \frac{1}{2(\rho-\kappa_1)} D^2\left(q_3\frac{dw}{dy} - q_2\frac{dv}{dz}\right) \ldots\ldots (1$$

with two others of like type.

These equations differ widely from those of Sarrau (see Gl brook's *Report*, p. 176); but they resemble those by which Boussi has explained the double refraction of quartz, if we put

$$q_1 = q_2 = q_3,$$

and, supposing these q's small, write

$$\frac{q_1}{\rho-\kappa_1} = \frac{q_2}{\rho-\kappa_2} = \frac{q_3}{\rho-\kappa_3},$$

i.e., neglect quantities of the order $\frac{q_1\kappa_1}{\rho^2}$ as small compared with We then have equations of the type

$$D^2u = F\frac{d\theta}{dx} + F_1\nabla^2 u + \frac{q}{\rho-\kappa_1}D^2\left(\frac{dw}{dy} - \frac{dv}{dz}\right) \ldots\ldots\ldots (li$$

where $$q_1/(\rho-\kappa_1) = q_2/(\rho-\kappa_2) = q/(\rho-\kappa_3),$$

at any rate to a first approximation.

This equality is not a necessity of Boussinesq's theory, any than the inequality is a necessity of the generalised equati Boussinesq's theory involves the inequality of κ_1, κ_2, and κ_3; but, if q's are equal and very small, the products of small quantities ma reasonably neglected and the above coefficients be all replaced by If any facts in the theory should require their inequality, they be retained in the form given above. I have not considered the mo of a wave in a medium for which equations (liii.) above in their gen form hold. With the equality of the twist coefficients and the ditions satisfied by the other coefficients [*i.e.*,

$$F/F_1 = G/G_1 = H/H_1,$$

and F, G, H differing only slightly among themselves, as also F_1, G_1, elliptic double refraction follows, just as in Boussinesq's investi tion.

17. *On Metallic Reflection.*

There are only *two* methods by means of which metallic reflection has been explained with any degree of success. In the first, the square of the refractive index r as obtained by equation (xxxiv.) of my Art 12 is supposed to be negative. This amounts to the theory of Cauchy developed by Eisenlohr,* and dealt with from an individual standpoint by Sir William Thomson.† In the second, of which Lord Rayleigh‡ may be taken as the best exponent, a differential equation involving a viscous term is adopted from a modified elastic solid theory. The latter explanation starts from an equation of the type

$$\rho \frac{d^2w}{dt^2} + h\frac{dw}{dt} = \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right) \quad \text{..................(liv.)},$$

for light polarised in the plane of incidence—where x is perpendicular to the plane of the reflecting surface, and $z = 0$ the trace of the wave front on that surface. It is also necessary to suppose ρ and h "subject to extensive chromatic variations" (*Phil. Mag.*, Vol. XLIII., p. 324). This leads to a value of r^2 containing a *positive* real and a *negative* imaginary part. According to Eisenlohr (cited by Glazebrook, p. 197), the real part of r^2 is for silver negative. So that the viscous-term explanation seems to fail at this point. Sir William Thomson, on the other hand, starts from the ordinary elastic equations, and, having found r^2 a negative quantity from his molecular theory, investigates the changes that take place in the ordinary formula [equations (xxxix.) and (xlvi.) above], for a transparent medium when r^2 is negative. The results obtained agree well with the facts until we have to deal with the small amount of chromatic dispersion observed in metallic reflection. To avoid the dispersion consequent on Thomson's theory, it is needful to suppose he holds the "effective rigidity of the ether in the interstices between the molecules" to be some improbable function of the period of vibration (see "Molecular Dynamics," p. 313). Lindemann (*Ueber Molekularphysik*, p. 9 of *Reprint*) endeavours to explain how special forms of the expression for r^2 (in the case of a *metal* with an immense number of spectral lines) may enable us to surmount Thomson's difficulty.

18. Turning now to the generalised theory of elasticity, we may ask the type of equation which it can provide for metallic reflection.

* See Glazebrook's *Report on Optical Theories*, p. 193 *et seq.*
† *Molecular Dynamics*, p. 307 *et seq.*
‡ *Phil. Mag.*, Vol. XLIII., 1872, p. 321 *et seq.*

It is difficult to conceive how a term of the form required by Lor Rayleigh ($h\,dw/dt$) could arise from the generalised expressions f the strain-energy. As it is only a differential with regard to t, it mu have arisen from the term

$$\frac{d}{dt}\left(\frac{d\phi}{d\dot{w}}\right),$$

but this involves a term of the type $-hw\dot{w}$ in ϕ, or the strai energy must be a function of the absolute displacements. But thi would denote that we must replace

$$\frac{d}{dt}\left(\frac{d\phi}{d\dot{w}}\right)$$

by the more complete operation

$$-\left(\frac{d\phi}{dw}-\frac{d}{dt}\frac{d\phi}{d\dot{w}}\right),$$

and this, applied to a term such as $-hw\dot{w}$, introduces no term wha ever into the generalised shift-equation. A viscous term, then, of th kind demanded by Lord Rayleigh, could not apparently be provide by the generalised theory of elasticity.

On the other hand, if we refer to the equations (xxxii.) or t (xliv.) of articles 12 and 13 above, we have

$$r^2 = r_0^2\left(\frac{1+\beta' c^2}{1+\beta'_1 c^2}\right) = \frac{\rho-\kappa_1}{\rho-\kappa}\left\{\frac{\mu+(\mu'+\gamma/4)\,c^2}{\mu+(\mu'_1+\gamma_1/4)\,c^2}\right\} \quad\text{.........(lv.}$$

Hence there are several possibilities of r^2 being made a negativ quantity. (It can obviously never be complex.)

(i.) For an opaque medium κ_1 may be greater than ρ, while for transparent medium κ may be smaller than ρ.

The difficulty of this hypothesis arises from the fact that, the velocit of a wave in a transparent medium being smaller than in the ether, w should expect for transparent bodies that the κ's would be essentiall *negative*. We should then have to assume that for an opaque mediur κ was of a totally different character to what it is for a transparen medium. This may, of course, be the fact which separates the tw classes of media physically, but there is no evidence, beyond the nee for explaining the phenomena of reflection by the theory, to induc us to suppose that κ is not always negative.

In this case, if μ'_1 and γ_1 were nearly equal to μ' and γ, there nee be very little chromatic dispersion.

(ii.) Suppose both κ and κ_1 negative, but that γ_1 is a large negative quantity. There are good reasons for supposing the γ's, like the κ's, to be negative, and the only assumption here will be that for an opaque medium γ_1 has a *large* negative value. Now Thomson and Lindemann develop r^2 in a series of powers of $1/c^2$. Thus we should find

$$r^2 = \frac{\rho-\kappa_1}{\rho-\kappa}\,\frac{\mu'+\gamma/4}{\mu_1'+\gamma_1/4}\left\{1+\frac{\mu}{c^2}\left(\frac{1}{\mu'+\gamma/4}-\frac{1}{\mu_1'+\gamma_1/4}\right)\right\} \quad \ldots\ldots \text{(lvi.)}$$

to the first power.

Hence, if γ_1 be $> -4\mu_1'$, r^2 will be negative, while, if there be chromatic dispersion due to the second term involving $1/c$, it does not seem improbable that γ_1 may bear such a relation to the other constants, that its effect will not be very sensible.

The generalised theory of elasticity appears able, then, to account for metallic reflection on the lines adopted by Sir William Thomson, on the very reasonable hypothesis that for an opaque body γ_1 is negative and greater than $4\mu_1'$, while for a transparent body γ is probably still negative, but less than $4\mu'$. This result is independent of any particular molecular hypothesis, and is based merely on the magnitude of one of the generalised elastic constants.

(iii.) We have seen that, to explain the rotation of the plane of polarisation, it is necessary to introduce into the expression for the strain-energy terms of the type ϕ_{24}, or the products of twists into speeds. If we take the plane $x = 0$ for the reflecting surface, and the axis of z for the trace upon it of the front of the incident wave, then for light polarised in the plane of incidence

$$u = v = 0, \text{ and } w \text{ is a function only of } x, y.$$

Hence the total strain-energy, since

$$\tau_{xz} = \frac{1}{2}\frac{dw}{dx} = \frac{1}{2}\sigma_{xz} \text{ and } \tau_{zy} = -\frac{1}{2}\frac{dw}{dy} = -\frac{1}{2}\sigma_{yz},$$

must be a function of

$$\dot{w},\quad \sigma_{xz},\quad \sigma_{yz},\quad \dot{\sigma}_{xz},\quad \dot{\sigma}_{yz},$$

or, so far as quadratic terms are concerned, must be of the form

$$\begin{aligned}\phi = {} & \frac{\kappa}{2}\dot{w}^2+\frac{\mu}{2}(\sigma_{yz}^2+\sigma_{xz}^2)+\frac{\nu}{2}(\dot{\sigma}_{yz}^2+\dot{\sigma}_{xz}^2)\\ & +\frac{\mu''}{2}(\sigma_{yz}\dot{\sigma}_{yz}+\sigma_{xz}\dot{\sigma}_{xz})+\frac{\mu''''}{2}(\dot{\sigma}_{yz}\sigma_{xz}+\dot{\sigma}_{xz}\sigma_{yz})\\ & +q\dot{w}(\sigma_{yz}+\sigma_{xz})+q'\dot{w}(\dot{\sigma}_{yz}+\dot{\sigma}_{xz}).\end{aligned}$$

We easily find

$$d\phi/d\dot{w} = \kappa\dot{w} + q\,(\sigma_{yz}+\sigma_{xz}) + q'\,(\dot{\sigma}_{yz}+\dot{\sigma}_{xz}),$$

$$\widehat{zx} = (\mu-\nu D^2)\,\sigma_{xz} + (q-q'D)\,\dot{w},$$

$$\widehat{zy} = (\mu-\nu D^2)\,\sigma_{yz} + (q-q'D)\,\dot{w}.$$

Thus the type of body-shift equation is

$$(\rho-\kappa)\,D^2w = 2q\,D\left(\frac{dw}{dy}+\frac{dw}{dx}\right) + (\mu-\nu D^2)\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}\right) \dots \text{(lvii.)}.$$

Thus we see that terms with the coefficients μ'', μ'''' and q' do not occur in this equation, and, so far as the determination of the refractive index is concerned, it is idle to consider the possibility of their occurrence in the value of ϕ. Indeed the terms in μ'' and μ'''' are of no importance, as they do not even enter into the stresses; while those, however, in q and q' introduce important terms into the shears, which have now terms (probably negative) proportional to the shift-speed. The existence of these terms does not seem beyond the range of possibility, and equation (lvii.) becomes that for the type of wave we are considering. Let us suppose that the term in q has a real existence for an opaque medium. Then we have the following equations for the two media:—

$$\left.\begin{aligned}(\rho_1-\kappa_1)\,\ddot{w}_1 &= 2q\,D\left(\frac{dw_1}{dy}+\frac{dw_1}{dx}\right) + (\mu_1-\nu_1 D^2)\left(\frac{d^2w_1}{dx^2}+\frac{d^2w_1}{dy^2}\right)\\ (\rho-\kappa)\,\ddot{w} &= (\mu-\nu D^2)\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}\right)\end{aligned}\right\}\text{..(lviii.)}.$$

Assume

$$w = Ae^{i(ax+by+ct)} + A'e^{i(-ax+by+ct)},$$

$$w_1 = Be^{i(a'x+by+ct)}.$$

We have at once

$$\left.\begin{aligned}(\rho_1-\kappa_1)\,c^2 &= 2qc\,(a'+b) + (\mu_1+\nu_1 c^2)(a'^2+b^2)\\ (\rho-\kappa)\,c^2 &= (\mu+\nu c^2)(a^2+b^2)\end{aligned}\right\}\text{............(lix.)}.$$

Thus

$$\frac{\rho_1-\kappa_1}{\rho-\kappa}(\mu+\nu c^2)(a^2+b^2) = 2qc\,(a'+b) + (\mu_1+\nu_1 c^2)(a'^2+b^2).$$

To explain metallic reflection, this equation ought to give us a complex value for a'. We may write the equation

$$a'^2 + 2Qa' + b^2 - L\,(a^2+b^2) + 2Qb = 0,$$

where $$Q = \frac{qc}{\mu_1+\nu_1 c^2}, \text{ and } L = \frac{\rho_1-\kappa_1}{\rho-\kappa}\,\frac{\mu+\nu c^2}{\mu_1+\nu_1 c^2}.$$

Thus we must have

$$Q^2 < b^2 - L(a^2+b^2) + 2Qb.$$

Or, for normal incidence when $b = 0$, we must have

$$Q^2 + La^2 < 0.$$

This is impossible unless L is negative; but, to make L negative, is equivalent to falling back on a negative ν_1, which is the second case we have considered above. Thus, to introduce terms of the form ϕ_{34} into the strain-energy does not give us any assistance in the problem of metallic reflection.

19. *Rotation of the Plane of Polarisation by Magnetism.*

Very different equations have been obtained by different writers for the motion of optical waves in a magnetic field.

Case (i.) Maxwell introduces a term into the kinetic energy for waves moving in the direction of the axis of z, of the form

$$C\gamma\left(\frac{d^2u}{dz^2}\dot{v} - \frac{d^2v}{dz^2}\dot{u}\right).$$

This in our notation may, in this case, be represented by

$$2C\gamma\left(\dot{v}\frac{d\tau_{zx}}{dz} - \dot{u}\frac{d\tau_{zy}}{dz}\right),$$

or $$C\gamma\left(\dot{v}\frac{d\sigma_{zx}}{dz} - \dot{u}\frac{d\sigma_{yz}}{dz}\right) \quad \text{.....................} \quad \text{(lx.)}.$$

Maxwell then states that the components of the impressed force deduced by Lagrange's equations will be of the type

$$X = \rho\frac{d^2u}{dt^2} - C\gamma\frac{d^3v}{dz^2dt} \quad \text{.....................} \quad \text{(lxi.)},$$

while those forces arising "from the action of the remainder of the medium on the element under consideration" must be in the case of an isotropic medium of the form indicated by Cauchy, or

$$X = A_0\frac{d^2u}{dz^2} + A_1\frac{d^4u}{dz^4} + \&c.$$

He thus arrives at equations of the type*

$$\rho\frac{d^2u}{dt^2}-C\gamma\frac{d^3v}{dz^2\,dt}=A_0\frac{d^2u}{dz^2}+A_1\frac{d^4u}{dz^4}+\&c.\ldots\ldots\ldots\ldots(\text{lxii.}).$$

There seems to me great difficulty in understanding exactly how and why the kinetic energy is thus to be separated from the strain-energy. It would be impossible on the theory of generalised elasticity. The effect of the angular velocity of an element of the medium would be to alter the intermolecular force, and this alteration of intermolecular force would affect the strain-energy. We should expect, therefore, the term above to occur in the value of the strain-energy ϕ; but then we must assume that this strain-energy depends not only on strain, shift-speeds, and strain-speeds, but also on the fluxions of the strain with regard to space. Now this dependence of the strain-energy on the strain-fluxions with regard to space, does not seem improbable in the case of magnetic action. In the particular molecular hypothesis to which I have previously referred, the magnetic terms introduce into the strain-energy terms *linear* in the shift-speeds, and so differing in character from the quadratic terms in which the strains usually appear. To a second approximation we should then find the products of shift-speeds and space-fluxions of the strain occurring in the value of the strain-energy.

In the particular case of motion we are dealing with, the only shifts are u and v, and the only strains σ_{xz} and σ_{yz}, equal respectively to du/dz and dv/dz. Thus the only possible terms which can arise in ϕ are of the form

$$\left.\begin{aligned}\phi_5 &= c_1\dot{u}\frac{d\sigma_{yz}}{dz}+c_1'\dot{u}\frac{d\sigma_{xz}}{dz}\\ &\quad+c_2\dot{v}\frac{d\sigma_{xz}}{dz}+c_2'\dot{v}\frac{d\sigma_{yz}}{dz}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(\text{lxiii.}).$$

Now let us discover what the value of

$$\iiiint\delta\phi_5\,dx\,dy\,dz\,dt$$

contributes to the new body-shift equations. We have it equal to

$$\iiiint\Big[c_1\left(\delta\dot{u}\frac{d^2v}{dz^2}+\dot{u}\frac{d^2\delta v}{dz^2}\right)+c_1'\left(\delta\dot{u}\frac{d^2u}{dz^2}+\dot{u}\frac{d^2\delta u}{dz^2}\right)$$

$$+c_2\left(\delta\dot{v}\frac{d^2u}{dz^2}+\dot{v}\frac{d^2\delta u}{dz^2}\right)+c_2'\left(\delta\dot{v}\frac{d^2v}{dz^2}+\dot{v}\frac{d^2\delta v}{dz^2}\right)\Big]\,dx\,dy\,dz\,dt.$$

* Art. 827, Vol. II., of the *Electricity and Magnetism*.

Integrating by parts, and retaining only quadruple integral terms, we obtain

$$\iiiint \left[c_1 \left(-\delta u \frac{d^3 v}{dz^2 dt} + \delta v \frac{d^3 u}{dz^2 dt} \right) + c_2 \left(-\delta v \frac{d^3 u}{dz^2 dt} + \delta u \frac{d^3 v}{dz^2 dt} \right) \right] dx\, dy\, dz\, dt.$$

Thus the constants c_1' and c_2' will not occur in the body-shift equations, and we need not further consider them.

We find, therefore, that

$$\phi_5 = c_1 \dot{u} \frac{d\sigma_{yz}}{dz} + c_2 \dot{v} \frac{d\sigma_{zx}}{dz} \quad \text{(lxiv.)}$$

introduces into the u- and v- shift-equations the terms

$$(c_2 - c_1) \frac{d^3 v}{dz^2 dt} \quad \text{and} \quad -(c_2 - c_1) \frac{d^3 u}{dz^2 dt}$$

on their right-hand sides respectively.

Now, in order that a rotation of the axes x, y round the axis of z may not affect the *form* of ϕ_5, it is needful that

$$c_2 = -c_1 = C\gamma, \text{ say.}$$

We thus reach for our strain-energy a form,

$$\phi_5 = C\gamma \left(\dot{v} \frac{d^2 u}{dz^2} - \dot{u} \frac{d^2 v}{dz^2} \right) \quad \text{(lxv.)},$$

identical with Maxwell's form for the kinetic energy of his system, but leading to just *double* of his expressions in the body-shift equations. These now take the type

$$(\rho - \kappa) \frac{d^2 u}{dt^2} = 2C\gamma \frac{d^3 v}{dz^2 dt} + (\mu - \mu''' D^2) \frac{d^2 u}{dz^2} \quad \text{(lxvi.)},$$

or, neglecting the dispersion terms, &c. (in κ and μ'''), we have

$$\left. \begin{aligned} \rho \frac{d^2 u}{dt^2} &= 2C\gamma \frac{d^3 v}{dz^2 dt} + \mu \frac{d^2 u}{dz^2} \\ \rho \frac{d^2 v}{dt^2} &= -2C\gamma \frac{d^3 u}{dz^2 dt} + \mu \frac{d^2 v}{dz^2} \end{aligned} \right\} \quad \text{(lxvii.)}.$$

The introduction of the factor 2 is thus the only difference which arises from dealing with the term which Maxwell takes as a part of his kinetic energy, as if it arose from the strain-energy in our generalised theory. (Similar equations follow from a more specialised atomic theory; see *Proc. London Math. Soc.*, Vol. xx., p. 62.)

Case (ii.) Lindemann (*Ueber Molekularphysik*, *Reprint*, p. 42) ha arrived at equations of the form

$$\left.\begin{aligned} \rho\frac{d^2u}{dt^2} &= \mu\frac{d^2u}{dz^2} + \nu\ (z_1 - u) - C\gamma\frac{d^3v}{dz\,dt^2} \\ \rho\frac{d^2v}{dt^2} &= \mu\frac{d^2v}{dz^2} + \nu'\ (z_1' - v) - C\gamma\frac{d^3u}{dz\,dt^2} \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(\text{lxviii.}$$

where $\frac{z_1}{u} = \frac{z_1'}{v}$ = a certain function of the period of vibration, an ν, ν' are constants. The generalised theory of elasticity does no obviously lend itself to the terms with coefficients ν and ν', and i seems in any case that they might be replaced by terms in d^2u/dt^2 an d^2v/dt^2, since the latter would only alter the above function of th period of vibration. Such terms arise, then, from the changes i apparent density of the medium produced by the κ-terms (ϕ_2) of ϕ the strain-energy. With regard to the last or rotatory terms, the differ essentially from Maxwell's, partly in being of like sign, an partly in the character of the differential. I do not see from wha terms in the generalised strain-energy they would be likely to arise nor do I follow the analysis by which Lindemann obtains them.

Case (iii.) Boussinesq* has deduced the magnetic rotation o the plane of polarisation by the introduction into the body-shif equations of the following terms, the wave moving parallel to th axis of z:—

$$\frac{d^3v}{dt^3} \text{ instead of Maxwell's } \frac{d^3v}{dz^2\,dt},$$

$$\text{and} \qquad \frac{d^3u}{dt^3} \quad \text{,,} \qquad \text{,,} \qquad \frac{d^3u}{dz^2\,dt}.$$

Maxwell does not seem to have seen Boussinesq's theory, for h remarks, "I am not aware that this form of the equations has bee suggested by any physical theory."† It gives a value for the angle o rotation which agrees tolerably well with observation for bodies o moderate dispersive power.

We may now ask whether our theory of generalised strain-energ can be made to render up Boussinesq's terms as well as Maxwell's Obviously, Boussinesq's theory cannot give terms like Maxwell's, a

* *Journal de Liouville*, T. XIII., 1868, pp. 430–433.

† *Electricity and Magnetism*, Art. 830, Vol. II., p. 411 (First Edition).

it must introduce second differentials with regard to t. It can, however, give Lindemann's terms, only they will appear with a *difference in sign*. These are, indeed, the terms by which on our theory of strain-energy we have endeavoured to explain the rotation of the plane of polarisation in a liquid.

In order to obtain Boussinesq's terms, we must suppose the strain-energy not only a function of the shift-speeds, but also of the shift-accelerations. Whether this assumption is probable or not, is another matter. Looked at from the standpoint of the particular molecular hypothesis, which leads me to believe that the strain-energy is a function not only of strain, but of shift- and strain-speeds, it does not seem impossible. Certain internal molecular coordinates have to be determined from equations involving molecular velocities, and the differentials of these quantities by the Hamiltonian method; hence the "constants" of intermolecular force depending on the internal molecular coordinates may really involve the mean molecular acceleration. The probability seems just as great as the assumption by Boussinesq that the shift of an element of the body through which the wave is passing is a function of the ether-velocity, as well as its shift at the same point.

We have, then, in the case where u and v are the only shifts, the following terms to be added to ϕ_2:—

$$k_1\dot{u}\ddot{v}+k_1'\dot{u}\ddot{u}+k_2\dot{v}\ddot{u}+k_2'\dot{v}\ddot{v}.$$

Hence the terms $\dfrac{d}{dt}\dfrac{d\phi}{d\dot{u}}$ and $\dfrac{d}{dt}\dfrac{d\phi}{d\dot{v}}$

in the body-shift equations must be replaced by

$$\frac{d}{dt}\left(\frac{d\phi}{d\dot{u}}\right)-\frac{d^2}{dt^2}\left(\frac{d\phi}{d\ddot{u}}\right)$$

and

$$\frac{d}{dt}\left(\frac{d\phi}{d\dot{v}}\right)-\frac{d^2}{dt^2}\left(\frac{d\phi}{d\ddot{v}}\right), \text{ respectively.}$$

This leads to terms $(k_1-k_2)\dddot{v}$ and $(k_2-k_1)\dddot{u}$, appearing in the u- and v- shift-equations respectively. Hence it is only necessary to deal with the terms

$$k_1\dot{u}\ddot{v}+k_2\dot{v}\ddot{u}$$

in the strain-energy; but, in order that a rotation of axes round z should leave these unchanged in form, we must have

$$k_1=-k_2=+C\gamma, \text{ say.}$$

Thus our equations of motion become

$$\left.\begin{aligned}(\rho-\kappa)\,\ddot{u} &= 2C\gamma\,\frac{d^3v}{dt^3}+(\mu-\mu'''D^2)\,\frac{d^2u}{dz^2}\\ (\rho-\kappa)\,\ddot{v} &= -2C\gamma\,\frac{d^3u}{dt^3}+(\mu-\mu'''D^2)\,\frac{d^2v}{dz^2}\end{aligned}\right\}\quad\ldots\ldots\ldots(\text{lxix.}).$$

These are of the form originally suggested by Airy (*Phil. Mag.*, June, 1846) to explain the phenomenon, and they have been fully discussed by Boussinesq in the memoir cited above.

Thus, while neither the Boussinesq nor the Maxwell terms alone will give results in complete accordance with experiment, it is possible that their combination, which is possible on the theory of generalised strain-energy, will be more successful. At any rate it is of value to see the form of terms in the strain-energy from which either set of equations arises; for it is to an expression for the generalised strain-energy that we may expect any future molecular theories to lead us.

19. *Aberration.*

We have seen (Eqn. xxvi.) that the generalised strain-energy leads, in the case of an isotropic elastic medium, to an equation of the form

$$\rho\,(\ddot{u}-X) = \kappa\ddot{u}+\{\overline{\lambda+\mu}-\overline{\lambda'''+\mu'''}D^2\}\,\frac{d\theta}{dx}+(\mu-\mu'''D^2)\,\nabla^2 u,$$

where $$D^2 = d^2/dt^2,$$

and $$\lambda''' = \lambda'-\gamma/2 \quad\text{and}\quad \mu''' = \mu+\gamma/4.$$

This equation is obtained on the hypothesis that the displacements are small, and we suppose the acceleration of the element $\ddot{u}$ on both sides of this equation to be the same. Now the term $\kappa\ddot{u}$ arises from the *strain-energy*, while the term $\rho\ddot{u}$ really expresses the so-called "effective force" per unit volume of the element. Hence $\ddot{u}$ in $\rho\ddot{u}$ has reference to axes fixed in space, while $\ddot{u}$ in $\kappa\ddot{u}$ is acceleration relative to the ether in the neighbourhood of the vibrating particles. Thus the d/dt in the terms $\rho\ddot{u}$ and $\kappa\ddot{u}$, when the medium is moving rapidly through space, will be different in character. But the difficulty is to know exactly what is motion relative to the ether. It would appear that a certain portion of the ether really moves with the transparent medium, while another portion appears to pass through the medium. If we attribute the light vibration to the

"bound" portion of the ether, then, by imposing on the system velocities equal and opposite to that with which the body is moving through space, $\ddot{u}$ in $\rho\ddot{u}$ will be the acceleration of the "bound" ether relative to the body, and $\ddot{u}$ in $\kappa\ddot{u}$ the acceleration of the "bound" ether relative to the free ether, which will now be passing through the bound ether with a velocity equal and opposite to that of the original velocity of the transparent medium. It seems, then, reasonable to replace the d/dt in $\kappa\ddot{u}$ by the form usual in hydro-dynamical investigations, as there is now no longer the condition of small relative velocities. If U, V, W be the velocity-components of the moving transparent medium, those of the free ether after the body has been brought to rest will be $-U$, $-V$, and $-W$. Now we shall suppose that U, V, W remain sensibly constant during a very great number of light vibrations. Hence, neglecting the products of shift-speeds, we shall have to replace $\dot{u}$ in $\kappa\ddot{u}$ by

$$\frac{du}{dt}+(-U)\frac{du}{dx}+(-V)\frac{du}{dy}+(-W)\frac{du}{dz},$$

or by
$$\left(\frac{d}{dt}-U\frac{d}{dx}-V\frac{d}{dy}-W\frac{d}{dz}\right)u.$$

Thus the equations of motion become of the type

$$\rho(\ddot{u}-X)=\kappa\left(\frac{d}{dt}-U\frac{d}{dx}-V\frac{d}{dy}-W\frac{d}{dz}\right)^2 u$$
$$+(\overline{\lambda+\mu}-\overline{\lambda'''+\mu'''}D^2)\frac{d\theta}{dx}+(\mu-\mu'''D^2)\nabla^2 u \ldots \text{(lxx.)}.$$

Let us take a wave of the form

$$\frac{u}{A}=\frac{v}{B}=\frac{w}{C}=e^{\sqrt{-1}(a_1x+a_2y+a_3z+c't)};$$

then
$$\frac{du}{dx}=\frac{a_1}{c'}\frac{du}{dt},\quad \frac{du}{dy}=\frac{a_2}{c'}\frac{du}{dt},\quad \frac{du}{dz}=\frac{a_3}{c'}\frac{du}{dt},$$

or
$$\frac{d}{dt}-U\frac{d}{dx}-V\frac{d}{dy}-W\frac{d}{dz}=\left(1-\frac{Ua_1+Va_2+Wa_3}{c'}\right)\frac{d}{dt}.$$

Thus, omitting X and the terms which give rise to dispersion, we have

$$\left\{\rho-\kappa\left(1-\frac{Ua_1+Va_2+Wa_3}{c'}\right)^2\right\}\frac{d^2u}{dt^2}=(\lambda+\mu)\frac{d\theta}{dx}+\mu\nabla^2 u \ldots \text{(lxxi.)}.$$

Now let α_1, β_1, γ_1 be the direction cosines of the wave-front, the velocity of the wave, then

$$\frac{U\alpha_1+V\alpha_2+W\alpha_3}{c'}=\frac{U\alpha_1+V\beta_1+W\gamma_1}{\Omega'}=\frac{\zeta}{\Omega'},$$

where $\zeta=$ velocity of medium resolved in direction of the Thus our equations of vibrations become of the type

$$\left\{\rho-\kappa\left(1-\frac{\zeta}{\Omega'}\right)^2\right\}\frac{d^2u}{dt^2}=\overline{\lambda+\mu}\,\frac{d\theta}{dx}+\mu\nabla^2u\,\ldots\ldots\ldots($$

These equations, therefore, lead to results precisely similar to the preceding articles, when the κ of those articles is replaced

$$\kappa\left(1-\frac{\zeta}{\Omega'}\right)^2,$$

κ being, as we have already noted, in all probability negative.

Thus the velocity of the wave of transverse vibrations given by

$$\Omega'^2=\frac{\mu}{\rho-\kappa\left(1-\dfrac{\zeta}{\Omega'}\right)^2}.$$

Let Ω be the velocity of the wave if $\zeta=0$. Then, if we negl squares of ζ/Ω' and put $\Omega=\Omega'$ in the small terms, we have

$$\Omega'^2=\Omega^2-2\Omega\frac{\kappa}{\rho-\kappa}\zeta,$$

or
$$\Omega'=\Omega-\frac{\kappa}{\rho-\kappa}\zeta.$$

Now, neglecting dispersion terms, and supposing $\kappa=0$, for th outside the transparent medium, we have for the refractive in

$$r^2=\frac{\rho-\kappa}{\rho},$$

Thus
$$1-\frac{1}{r^2}=-\frac{\kappa}{\rho-\kappa},$$

or, finally,
$$\Omega'=\Omega+\left(1-\frac{1}{r^2}\right)\zeta\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1$$

This is the formula given by Fresnel for aberration, and confir

experiment. It has been obtained by neglecting the dispersion terms, and supposing the transparent medium to be in motion relative to the free ether. The type of equation we have obtained, and the method in which I have dealt with it, are both the same as those of Boussinesq,* but the assumptions on which our equations are based are very different. My equations are here, as elsewhere, obtained from the assumption that the strain-energy is a function not only of the strain, but of the strain-speeds, and of the shift-speeds relative to the free ether. I have avoided, as far as possible, making any assumptions as to the medium which propagates light in material bodies; I have not definitely identified it with the "bound" ether, or with the matter of the transparent body, or with a combination of the two. The sole hypothesis is, that it is a medium for which the generalised equations of elasticity hold, and that this medium moves relatively to the free ether. I think Boussinesq supposes the whole of the ether to be in motion† through the body, and he also considers that intermolecular distance is very great as compared with molecular diameter.‡ Boussinesq's theory is, therefore, not a purely elastic medium theory, as the present theory professes to be.

20. We have seen in the present paper how far an elastic medium theory can be extended in the endeavour to explain the phenomena of light. With the generalised equations of elasticity, the terms which give rise to dispersion in a simple transparent medium enter naturally, but anomalous dispersion and absorption are beyond the control of such a theory; the rotation of the plane of polarised light by certain liquids is explained by the assumption of "dissymmetrical isotropy" in such media; the terms involving rotation of the plane of polarised light by magnetism are obtained also from the generalised strain-energy, and these terms may combine both the sets to which Maxwell and Boussinesq have attributed the rotation. The phenomena of double refraction and elliptic refraction, as well as those of ordinary refraction and reflection, are explained better than by the

* See the *Journal de Liouville*, Vol. XIII., 1868, pp. 433–438.

† "Comme les actions développées entre l'éther et la matière pondérable, lors de tout mouvement d'amplitude finie, sont extrêmement faibles, l'éther du corps ne sera presque pas entraîné avec lui, mais restera comparativement immobile." —*Liouville*, XIII., 1868, p. 433.

‡ Compare with the whole of the above a second paper of Boussinesq's on aberration, *Journal de Liouville*, Vol. XVIII., 1873, pp. 361–390, where he supposes the observer, the transparent body, and the source of light have no relative motion, but a common velocity relative to the ether.

ordinary theory of elasticity, but with the same sort of difficulties arise in the wave theory of Boussinesq. The present theory seems al to throw a considerable amount of light on elastic after-strain, a shows that the velocity of sound in bars will vary with the length the wave.

In another paper I propose to explain how the terms in th generalised strain-energy arise in the case of at least one molecul hypothesis.

Thursday, June 13th, 1889.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

The following communications were made:—

The Square of Euler's Series: Dr. Glaisher, F.R.S.

A Theorem in the Calculus of Linear Partial Differential Opera tions: Major P. A. Macmahon, R.A.

On Crystalline Reflection and Refraction: A. B. Basset, F.R.S.

On some Rings of Circles connected with a Triangle and th Circles (Schoute's system) that cut them at Equal Angles W. W. Taylor, M.A.

The Figures of the Pippian and Quippian of a Class of Plan Cubics: The President (Sir J. Cockle in the chair).

Generalization of Buffon's Problem: Dr. Sylvester, F.R.S (communicated by Mr. J. Hammond).

On the Small Wave-Motions of a Heterogeneous Fluid unde Gravity: Prof. W. Burnside, M.A.

On the Uniform Deformation in Two Dimensions of a Cylindrica Shell of Finite Thickness, with Applications to the Genera Theory of Deformation of Thin Shells: Lord Rayleigh Sec. R.S.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLV., Nos. 278, 279.

"An Elementary Treatise on Dynamics," by B. Williamson and F. A. Tarleton 8vo; London, 1889.

"Mathematics from the 'Educational Times,'" Vol. L.

"Educational Times," for June.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIII., Stück 4 Leipzig, 1889.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXIII., Livr. 2me; Harlem, 1889.

"Bulletin des Sciences Mathématiques," Tome XIII., May, 1889.

"Bulletin de la Société Mathématique de France," Tome XVII., No. 1.

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo II., No. 7; Mexico, 1889.

"Annali di Matematica," Tomo XVII., Fasc. 1; May, 1889.

"Journal of the College of Science, Imperial University, Japan," Vol. II., Part V.; Tokio, 1889.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Num. 81, 82.

"Tavola Sinottica delle Pubblicazioni Italiane registrate nel Bollettino della Biblioteca Nazionale Centrale di Firenze che furono ricevute dalle altre Biblioteche Pubbliche Governative Italiane nel 1888."

"Rendiconti del Circolo Matematico di Palermo," Tomo III., Fasc. II., April 1889; Palermo.

"Journal für die reine und angewandte Mathematik," Band 104, Heft IV.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig—Mathematisch-Physische Classe," 1889, I.; Leipzig.

Four pamphlets by F. Klein:—"Des fonctions thêta sur la surface générale de Riemann." "Formes principales sur les surfaces de Riemann." "Zur Theorie der Abel'schen Functionen." "Zur Theorie der Abel'schen Functionen" (Zweite Mittheilung).

"Sur le développement des fonctions implicites," par M. F. Gomes Teixeira; 4to pamphlet (from "Journal de Mathématiques pures et appliquées," Paris, 1888).

"Beiträge zur Kenntniss der Oxydationsvorgänge in Liebenden Zellen," von Dr. W. Pfeffer.

"Proceedings of the Canadian Institute," Toronto, Third Series, Vol. VI., Fasc. No. 2, April 1889; Toronto.

"Annual Report of the Canadian Institute, Session 1887–8" (part of Appendix I. to the Report of the Minister of Education, Ontario, 1888); Toronto, 1889.

"Annales de l'Ecole Polytechnique de Delft," Tome IV., 4e Livraison, 1888; Leide, 1888.

On Crystalline Reflection and Refraction. *By* A. B. Basset.

[*Read June 13th*, 1889.]

1. The electro-magnetic theory of light has been applied to the problem of reflection and refraction by the authors cited below;* and the general equations which determine the intensities of the two

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* Lorentz, *Schlömilch Zeitschrift*, Vol. XXII.; J. J. Thomson, *Phil. Mag.*, April, 1880; Fitzgerald, *Phil. Trans.*, 1880; Lord Rayleigh, *Phil. Mag.*, August, 1881; *Ibid.*, September, 1888.

refracted waves and the reflected wave, and the deviation of the plane of polarisation of the latter, when polarised light is incident upon the surface of a doubly refracting crystal, have been given by Glazebrook.* It is remarkable that these equations are of the same form as those obtained as long ago as 1835 by MacCullagh, by means of an erroneous theory. MacCullagh published two papers on the subject of "Crystalline Reflection and Refraction." In the first paper† he attempted to solve the problem by means of a generalisation of the assumptions employed by Fresnel in the case of isotropic media. His four assumptions are the following :—(i.) that the displacements are continuous; (ii.) that the kinetic energy per wave-length of the incident wave is equal to the kinetic energy per wave-length of the reflected and refracted waves; (iii.) that the density of the ether is the same in all media; (iv.) that the vibrations of polarised light are parallel to the plane of polarisation. Probably little fault will be found with the first two assumptions, but the last two are not generally accepted as correct at the present time. In his second paper,‡ MacCullagh attempted to deduce his previous results by means of a dynamical theory, which consisted in supposing that the potential energy per unit of volume of the ether within a crystal is a quadratic function of the rotations ξ, η, ζ; where $\xi = dw/dy - dv/dz$, &c., u, v and w being the displacements. This assumption as to the form of the energy of an anisotropic elastic medium, is well known to be erroneous, the correct expression having been given by Green; and therefore MacCullagh's final equations for determining the intensities of the reflected and refracted light, cannot be considered to stand upon a sound dynamical basis; but, as these final equations are the same as those furnished by the electro-magnetic theory, most of the results of MacCullagh's first paper, with certain modifications necessitated by his having supposed that the vibrations of polarised light are parallel to the plane of polarisation, are applicable to the latter theory, and thus MacCullagh's results regain their interest.

A theory totally different from the electro-magnetic theory has been recently proposed by Sir W. Thomson,§ which, so far as it applies to isotropic media, consists in regarding the luminiferous ether as an elastic medium, whose elasticity of volume is negative, and equal, or very nearly so, to $-\frac{4}{3}n$, where n is the rigidity; and he has applied this theory to the problem of reflection and refraction at the common surface of two isotropic media. This theory, by the aid of a

* *Proc. Camb. Phil. Soc.*, Vol. IV., p. 155.

† *Trans. Roy. Irish Acad.*, Vol. XVIII., p. 31.

‡ *Trans. Roy. Irish Acad.*, Vol. XXI., p. 17.

§ *Phil. Mag.*, November and December, 1888.

subsidiary hypothesis, which was originally suggested by Rankine and Stokes,* and more fully developed by Lord Rayleigh,† has been applied by Glazebrook‡ to the problem of double refraction; and it is found to lead to Fresnel's wave-surface, and also makes the vibrations of a ray of polarised light, on emerging from a crystal, perpendicular to the plane of polarisation, although the direction of vibration within the crystal is not quite the same as that given by Fresnel and the electro-magnetic theory.

In the present paper I propose to discuss some of the results of the electro-magnetic theory, with special reference to reflection and refraction at the surface of a uniaxal crystal and to MacCullagh's work; in the next place, I shall show how Green's theory fails when it is applied to crystalline reflection and refraction; and lastly, I shall solve the problem by means of Sir W. Thomson's theory.

The Electro-magnetic Theory.

2. Let i be the angle of incidence; r_1, r_2 the angles which the two refracted wave normals make with the normal to the reflecting surface; A, A', A_1, A_2 the square roots of the intensities of the incident, reflected, and two refracted waves; let θ, θ', θ_1, θ_2 be the angles which the directions of vibration in these four waves make with their common line of intersection with the reflecting surface; also let χ_1, χ_2 be the angles between the two refracted rays and the corresponding wave normals.

If we suppose that the magnetic inductive capacity is the same in the two media (which is very nearly the case for all transparent dielectrics), Glazebrook's equations§ may be put into the form

$$(A\cos\theta + A'\cos\theta')\sin i = A_1\cos\theta_1\sin r_1 + A_2\cos\theta_2\sin r_2 \quad\ldots\ldots (1),$$

$$(A\cos\theta - A'\cos\theta')\cos i = A_1\cos\theta_1\cos r_1 + A_2\cos\theta_2\cos r_2 \quad\ldots\ldots (2),$$

$$A\sin\theta + A'\sin\theta' = A_1\sin\theta_1 + A_2\sin\theta_2 \quad\ldots\ldots (3),$$

$$(A\sin\theta - A'\sin\theta')\sin 2i = A_1(\sin\theta_1\sin 2r_1 + 2\sin^2 r_1\tan\chi_1) + A_2(\sin\theta_2\sin 2r_2 + 2\sin^2 r_2\tan\chi_2) \ldots (4).$$

3. MacCullagh supposed that the vibrations of polarised light

* *Brit. Assoc. Rep.* on "Double Refraction," 1862.

† Hon. J. W. Strutt, *Phil. Mag.*, June, 1871.

‡ *Ibid.*, December, 1888. [October, 1889. Since this paper was read, a paper has been published by Mr. Glazebrook in the *Philosophical Magazine* for August, 1889, in which he has applied Sir W. Thomson's theory to the problem of "Crystalline Reflection and Refraction," and has obtained the same results as are given in the present paper.]

§ *Proc. Camb. Phil. Soc.*, Vol. IV., p. 165.

were parallel to, instead of perpendicular to, the plane of polarisation; but his geometrical theorems with regard to uniradial directions may be easily modified so as to apply to the present theory. When polarised light is incident upon a crystalline plate at a given angle, it is always possible, by properly choosing the plane of polarisation, to make one or other of the two refracted rays disappear. The two directions of vibration for which this is possible are called *uniradial directions*. We can thus obtain the following modification of a theorem due to MacCullagh, viz.: that whenever one of the refracted rays is absent, the lines of intersection of the planes of polarisation of the three waves with their respective wave fronts lie in a plane.

Let the axis of x be the normal to the reflecting surface, and let the line of intersection of the latter with the wave fronts be the axis of z; O the point of incidence, OP, OP', OP_1 the lines of intersection of the three planes of polarisation with their respective wave fronts; λ, μ, ν; λ', μ', ν'; λ_1, μ_1, ν_1 the direction cosines of OP, OP', OP_1.

Then

$$\lambda = \cos\theta \sin i, \quad \mu = \cos\theta\cos i, \quad \nu = -\sin\theta,$$

$$\lambda' = \cos\theta' \sin i, \quad \mu' = -\cos\theta'\cos i, \quad \nu' = -\sin\theta',$$

$$\lambda_1 = \cos\theta_1 \sin r_1, \quad \mu_1 = \cos\theta\cos r_1, \quad \nu_1 = -\sin\theta_1.$$

Putting $A_2 = 0$ in (1), (2), (3), and substituting, we obtain

$$A\lambda + A'\lambda' = A_1\lambda_1,$$

$$A\mu + A'\mu' = A_1\mu_1,$$

$$A\nu + A'\nu' = A_1\nu_1;$$

whence
$$\begin{vmatrix} \lambda, & \lambda', & \lambda_1 \\ \mu, & \mu', & \mu_1 \\ \nu, & \nu', & \nu_1 \end{vmatrix} = 0,$$

which is the condition that OP, OP', OP_1 should lie in the same plane.

4. The intensities of the reflected and refracted waves, when the second medium is a uniaxal crystal which is cut perpendicularly to its axis, may be deduced from the general formulæ (1), (2), (3), and (4); but in this case it is simpler to proceed from first principles.

The wave-surface consists of the sphere

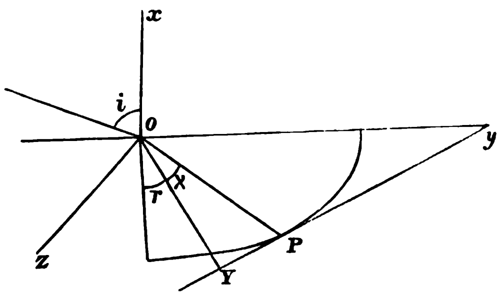
$$x^2 + y^2 + z^2 = c^2,$$

and the ellipsoid

$$x^2/c^2+(y^2+z^2)/a^2=1,$$

and if the crystal is a negative one, such as Iceland spar, $a > c$, so that the ellipsoid is planetary.

Let us first suppose that the incident light is polarised perpendicularly to the plane of incidence, so that the refracted ray is an extraordinary ray; and let S, S', S_1 be the electric displacements in the incident, reflected, and refracted waves.



The condition that the electric displacements perpendicular to the reflecting surface should be the same in both media, gives

$$(S+S')\sin i=-S_1\sin r \quad\text{.......................}(5).$$

The condition that the electric forces parallel to the reflecting surface should be the same in both media, gives

$$V^2(S-S')\cos i=-S_1 c^2\cos r \quad\text{.......................}(6),$$

where V is the velocity of light in the first medium.

If A, A', A_1 be the square roots of the intensities,

$$\frac{A}{VS}=\frac{A'}{VS'}=\frac{A_1}{V_1 S_1} \quad\text{.........................}(7).$$

Also,
$$\left.\begin{aligned} V/V_1&=\sin i/\sin r \\ V_1^2&=c^2\cos^2 r+a^2\sin^2 r=p^2 \end{aligned}\right\} \quad\text{..............}(8),$$

where p is the perpendicular from the point of incidence on to the tangent plane to the ellipsoid at the extremity of the extraordinary ray. Equations (5) and (6), therefore, become

$$A+A'=-A_1 \quad\text{..}(9),$$

$$A-A'=-\frac{A_1 c^2\sin 2r}{(a^2\sin^2 r+c^2\cos^2 r)\sin 2i} \quad\text{.................}(10),$$

and therefore

$$A' = A\frac{p^2 \sin 2i - c^2 \sin 2r}{p^2 \sin 2i + c^2 \sin 2r} \quad \ldots\ldots\ldots\ldots\ldots$$

$$A_1 = \frac{2A\,p^2 \sin 2i}{p^2 \sin 2i + c^2 \sin 2r} \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

which determine the intensities of the reflected and refracted v

If OP be the extraordinary ray, OY the perpendicular negative sign shows that the vibration in the extraordinary wav the direction YP.

The intensity of the reflected light vanishes when

$$p^2 \sin 2i - c^2 \sin 2r = 0,$$

and therefore, by (8),

$$V^2 \cot i = c^2 \cot r,$$

whence, eliminating r, we obtain

$$\tan i = \frac{V\sqrt{(V^2-c^2)}}{c\sqrt{(V^2-a^2)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

If, therefore, common light be incident upon the crystalline pl this angle, the reflected light will be completely polarised in the of incidence; (13), therefore, determines the polarising angle.

5. Let us now suppose that the incident light is polarised i plane of incidence.

The condition that the electric forces parallel to the pla incidence should be continuous, gives

$$V^2(S+S') = c^2 S_1;$$

and the condition that the corresponding components of the ma forces should be continuous, gives

$$(S-S')\,V\cos i = cS_1 \cos r,$$

whence

$$(A+A')\sin i = A_1 \sin r,$$

$$(A-A')\cos i = A_1 \cos r,$$

and therefore

$$A' = -\frac{A\sin(i-r)}{\sin(i+r)},$$

$$A_1 = \frac{A\sin 2i}{\sin(i+r)}.$$

These are the same formulæ as are known to hold good in the

of two isotropic media, which might have been expected, since the law of refraction of the ordinary ray is the same as in the case of isotropic media.

6. When light is internally reflected at the surface of glass which is in contact with a rarer medium, such as air, it is known that total reflection is accompanied by a change of phase; and since the results which hold good in the case of two isotropic media are particular cases of more general results which hold good in the case of crystalline media, it follows that there must be a change of phase when a crystal is substituted for the glass.

If the incident light is polarised perpendicularly to the plane of incidence, (9) and (10) must be replaced by

$$A+A' = A_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

$$A-A' = A_1\,(a^2\sin^2 i+c^2\cos^2 i)\sin 2r\,/\,c^2\sin 2i \quad\ldots\ldots\ldots\ldots(15),$$

whence
$$A' = A\,\frac{c^2\sin 2i-(a^2\sin^2 i+c^2\cos^2 i)\sin 2r}{c^2\sin 2i+(a^2\sin^2 i+c^2\cos^2 i)\sin 2r} \quad\ldots\ldots\ldots\ldots(16).$$

In these formulæ, the incident wave is the real part of

$$(A/V_1)\,\epsilon^{\iota\kappa'(x\cos i-y\sin i+V_1 t)},$$

where $2\pi/\kappa'$ is the wave-length, and V_1 the velocity of the incident wave in the crystal; and the reflected and refracted waves are the real parts of

$$(A'/V_1)\,\epsilon^{\iota\kappa'(-x\cos i-y\sin i+V_1 t)} \text{ and } (A_1/V)\,\epsilon^{\iota\kappa(x\cos r-y\sin r+Vt)}.$$

Since
$$a^2\sin^2 i+c^2\cos^2 i = V_1^2 = V^2\sin^2 i/\sin^2 r,$$

it follows that
$$\cos^2 r = \frac{c^2\cos^2 i-(V^2-a^2)\sin^2 i}{a^2\sin^2 i+c^2\cos^2 i} \quad\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

and therefore, since $V > a$, it follows that $\cos r$ becomes imaginary when

$$\tan i > \frac{c}{\sqrt{(V^2-a^2)}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

This is therefore the value of the critical angle at which total reflection takes place.

In order to calculate the change of phase which occurs, let

$$A' = \alpha+\iota\beta, \quad A_1 = \alpha_1+\iota\beta_1,$$

where a, β, a_1, β_1 are real. Then (14) and (15) become

$$A+a+\iota\beta = a_1+\iota\beta_1,$$

$$A-a-\iota\beta = \iota\,(a_1+\iota\beta_1)\,q,$$

where
$$q = V\left\{(V^2-a^2)\tan^2 i - c^2\right\}^{\frac{1}{2}}/c^2,$$

by (17). Whence
$$A+a = a_1, \qquad \beta = \beta_1,$$

$$A-a = -q\beta_1, \quad \beta = -qa_1;$$

accordingly, $a = \dfrac{A\,(1-q^2)}{1+q^2}$, $\beta = -\dfrac{2Aq}{1+q^2}$, $a' = \dfrac{2A}{1+q^2}$.

The reflected wave is therefore

$$(a/V_1)\cos\frac{2\pi}{\lambda'}(x\cos i+y\sin i-V_1t)+(\beta/V_1)\sin\frac{2\pi}{\lambda'}(x\cos i+y\sin i-V_1t),$$

or
$$(A/V_1)\cos\left\{\frac{2\pi}{\lambda'}(x\cos i+y\sin i-V_1t)+2e\right\},$$

where
$$\tan 2e = \frac{2q}{1-q^2};$$

whence
$$\tan e = q = V\left\{(V^2-a^2)\tan^2 i-c^2\right\}^{\frac{1}{2}}/c^2 \quad\ldots\ldots\ldots\ldots\ldots(19),$$

which determines the change of phase.

In order to obtain the corresponding result for an isotropic medium, we must put $a = c$, $V/a = \mu$, where μ is the index of refraction of the two media, and (19) becomes

$$\tan e = \mu\sqrt{(\mu^2\tan^2 i-\sec^2 i)},$$

which is the same expression as that obtained by Fresnel, and which approximately agrees with that obtained by Green on the theory of an elastic medium.

In the case of light polarised in the plane of incidence, it can be shown in a similar manner that

$$\tan e = \frac{\sqrt{(\mu^2\sin^2 i-1)}}{\mu\cos i},$$

which is Fresnel's and Green's result.

7. When the reflecting surface is parallel to the axis of the crystal, results of corresponding simplicity can be obtained, provided the plane of incidence contains the axis.

For light polarised perpendicularly to the plane of incidence, the

formulæ can easily be shown to be

$$A+A' = A_1,$$

$$A-A' = \frac{A_1 a^2 \sin 2r}{(a^2 \cos^2 r + c^2 \sin^2 r) \sin 2i}.$$

These equations are what equations (9) and (10) become when a and c are interchanged, and the sign of A_1 is changed. Hence the value of the polarising angle, the angle at which total reflection takes place, and the change of phase are obtained for this case by writing a for c in (13), (18), and (19).

8. There would be no difficulty in working out the results when the plane of incidence contains the axis, but the axis does not lie in the reflecting surface, although the results would be rather long. The value of the polarising angle may, however, be directly obtained from (1), (2), (3), and (4), by putting

$$A' = A_2 = 0, \quad \theta = \theta' = \theta_1 = \tfrac{1}{2}\pi,$$

eliminating A and A_1, and remembering that

$$\tan \chi_1 = \frac{(a^2-c^2) \sin \omega \cos \omega}{a^2 \sin^2 \omega + c^2 \cos^2 \omega},$$

where ω is the angle which the extraordinary wave normal makes with the optic axis. This is what has been done by MacCullagh, and his result is, that if λ be the angle which the optic axis makes with the reflecting surface, the polarising angle i is given by the elegant formula

$$\sin^2 i = \frac{V^2 (V^2 - a^2 \cos^2 \lambda - c^2 \sin^2 \lambda)}{V^4 - a^2 c^2}.$$

This value of the polarising angle (which MacCullagh states was first obtained by Seebeck) is easily seen to agree with the particular results which we have obtained for the cases $\lambda = 0$, $\lambda = \frac{1}{2}\pi$.

Green's Theory.

9. We shall now very briefly consider the application of Green's theory.

If an anisotropic medium be symmetrical with respect to three fixed rectangular planes, the potential energy will be of the form

$$W = \tfrac{1}{2} (Ee^2 + Ff^2 + Gg^2 + 2E'fg + 2F'ge + 2G'ef + Aa^2 + Bb^2 + Cc^2),$$

where e, f, g, a, b, c denote the strains.

In a medium of this kind it is not possible for distortional waves

to be propagated independently of dilatational waves, unless certain relations exist between the coefficients, which reduce W to the form

$$W = \tfrac{1}{2}\mu\,(e+f+g)^2 + \tfrac{1}{2}A\,(a^2-4fg) + \tfrac{1}{2}B\,(b^2-4ge) + \tfrac{1}{2}C\,(c^2-4ef),$$

which requires that

$$\left.\begin{array}{l} E = F = G = \mu \\ E' = \mu-2A, \quad F' = \mu-2B, \quad G' = \mu-2C \end{array}\right\} \quad \text{.........(19a).}$$

Green therefore assumes that these relations exist amongst the coefficients, and thus the equations of motion reduce to the form

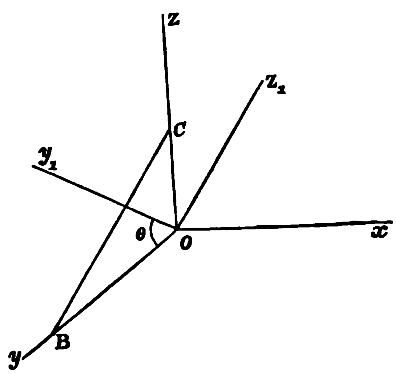
$$\rho\,\frac{d^2u}{dt^2} = \mu\,\frac{d\delta}{dx} + B\,\frac{d\eta}{dz} - C\,\frac{d\zeta}{dy},$$

with two similar ones, where δ is the dilatation and ξ, η, ζ are the rotations.

It is to be noticed that, if $\mu = 0$, the equations satisfied by the displacements in Green's theory are of the same form as those satisfied by the magnetic displacements in the electro-magnetic theory; whilst the equations which are satisfied by the rotations ξ, η, ζ are the same as those satisfied by the electric displacements.*

10. Green's medium also possesses the following property.

Let Ox, Oy, Oz be the axes of crystalline symmetry; and let BC be the intersection of any plane parallel to Ox with the plane yz; and



consider a portion of the medium which is bounded by the plane BC and two fixed rigid planes perpendicular to Ox. Draw Oy_1, Oz_1 re-

* Glazebrook, *Proc. Camb. Phil. Soc.*, Vol. IV., pp. 157 and 159.

spectively perpendicular and parallel to BC, and let the suffixed letters denote the values of corresponding quantities referred to Ox, Oy_1, Oz_1 as axes.

If in a crystalline medium which possesses three rectangular planes of symmetry, a shearing stress S_1 be applied to the plane BC, this stress will usually be a function of the extensions along Oy_1 and Oz_1, as well as of the shearing strain a_1, unless the plane BC be a principal plane; but, in a medium such as Green considers, the coefficients of the extensions are zero, and the relation $S_1 = Aa_1$ holds good for all positions of BC.

In order to prove this, we see at once that

$$S_1 = S\cos 2\theta + \tfrac{1}{2}(R-Q)\sin 2\theta.$$

Also, since the medium is supposed to be bounded by two rigid planes perpendicular to Ox, there can be no extension nor contraction parallel to Ox, whence

$$Q = Ff + E'g, \quad R = E'f + Gg;$$

accordingly,

$$S_1 = Aa\cos 2\theta + \tfrac{1}{2}\left\{(E'-F)f + (G-E')g\right\}\sin 2\theta \quad\ldots\ldots(19b).$$

But, if $m = \cos\theta$, $n = \sin\theta$,

$$f = \left(m\frac{d}{dy_1} - n\frac{d}{dz_1}\right)(mv_1 - nw_1)$$

$$= m^2 f_1 + n^2 g_1 - mna_1;$$

also $$g = n^2 f_1 + m^2 g_1 + mna_1.$$

Again,

$$a = \frac{dw}{dy} + \frac{dv}{dz}$$

$$= \left(m\frac{d}{dy_1} - n\frac{d}{dz_1}\right)(nv_1 + mw_1) + \left(n\frac{d}{dy_1} + m\frac{d}{dz_1}\right)(mv_1 - nw_1)$$

$$= a_1\cos 2\theta + (f_1 - g_1)\sin 2\theta.$$

Substituting in (19*b*), we obtain

$$S_1 = a_1\left\{A\cos^2 2\theta + \tfrac{1}{4}(G+F-2E')\sin^2 2\theta\right\}$$

$$+ \tfrac{1}{2}(f_1 - g_1)\left\{A - \tfrac{1}{4}(G+F-2E')\right\}\sin 4\theta$$

$$+ \tfrac{1}{4}(f_1 + g_1)(G-F)\sin 2\theta.$$

It therefore follows that if

$$G = F = \mu,$$
$$A = \tfrac{1}{2}(G+F-2E') = \tfrac{1}{2}(\mu - E'),$$

we shall have
$$S_1 = Aa_1.$$

A similar property is also true in the case of each of the other two axes; hence, if any two of the coordinate axes be rotated round the third, the shearing stress which tends to produce rotation about the third axis, is always equal to the product of the shearing strain and the principal rigidity corresponding to that axis.

It therefore follows that the relations between the constants which have been assumed by Green, are not mere arbitrary assumptions which have been made for the purpose of obtaining a particular analytical result, but correspond to definite physical properties of the medium.

11. The theory of Green, although dynamically sound, renders it necessary to suppose that the vibrations of polarised light are parallel to the plane of polarisation, which is one objection; also, if we agree to disregard this difficulty, another difficulty crops up in applying the theory to crystalline reflection and refraction, owing to the necessity of making some assumption involving relations between the physical constants of isotropic and crystalline media.

To investigate this point, let us consider the reflection and refraction of light at the surface of a uniaxal crystal, whose face is perpendicular to the axis. In order that the incident light should give rise to an extraordinary wave, it is necessary, on this theory, to suppose that the incident vibrations are perpendicular to the plane of incidence.

In the first medium, the equation of motion is

$$\rho \frac{d^2w}{dt^2} = n\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right) \quad \text{......................(20)},$$

and in the crystal
$$\rho_1 \frac{d^2w_1}{dt^2} = a^2 \frac{d^2w_1}{dy^2} + c^2 \frac{d^2w_1}{dx^2} \quad \text{......................(21)},$$

where we have written a^2, c^2 for A and C.

Let
$$w = S\,\epsilon^{\iota\kappa(x\cos i - y\sin i + Vt)} + S'\epsilon^{\iota\kappa(-x\cos i - y\sin i + Vt)},$$
$$w_1 = S_1\epsilon^{\iota\kappa_1(x\cos r - y\sin r + V_1 t)},$$

where
$$\kappa\sin i = \kappa_1 \sin r, \quad \kappa V = \kappa_1 V_1 \quad \text{..................(22)}.$$

From (20) we obtain $V^2 = n/\rho$,

and from (21) $V_1^2 = (a^2 \sin^2 r + c^2 \cos^2 r)/\rho_1$.

The surface conditions for continuity of displacement and stress give

$$w = w_1, \quad n\frac{dw}{dx} = c^2 \frac{dw_1}{dx},$$

when $x = 0$; whence $S + S' = S_1$,

$$\kappa n (S - S') \cos i = S_1 c^2 \kappa_1 \cos r,$$

the last of which, by (22), becomes

$$S - S' = S_1 \frac{c^2 \tan i}{n \tan r},$$

whence

$$S' = S\frac{n \tan r - c^2 \tan i}{n \tan r + c^2 \tan i} \quad \text{.........................(23)},$$

$$S_1 = \frac{2Sn \tan r}{n \tan r + c^2 \tan i} \quad \text{.........................(24)}.$$

We have hitherto avoided assuming that any relations exist between the physical constants of the two media; but, in order that these results should be consistent with those which the theory furnishes for isotropic media, it seems necessary to suppose that $n = c^2$. Now the intensity of light is measured by the energy, and the formulæ then show that the intensity of the reflected light would be the same as if the crystal were an isotropic medium, whilst that of the refracted light would be different. These results are, however, inconsistent with the electro-magnetic theory. Since the wave whose velocity is c is refracted according to the ordinary law, the assumption that $n = c^2$ might at first sight appear to be a plausible one in the case of uniaxal crystals; but, if we attempt to apply the theory to biaxal crystals, there is no valid reason why n should be assumed to be equal to one of the three principal rigidities, rather than to either of the other two.

If we adopt the assumption of MacCullagh and Neumann, that $\rho = \rho_1$, the intensities will be proportional to the square roots of the amplitudes, and we shall obtain

$$A' = A\frac{(a^2 \sin^2 r + c^2 \cos^2 r) \sin 2i - c^2 \sin 2r}{(a^2 \sin^2 r + c^2 \cos^2 r) \sin 2i + c^2 \sin 2r},$$

$$A_1 = \frac{2A\,(a^2 \sin^2 r + c^2 \cos^2 r) \sin 2i}{(a^2 \sin^2 r + c^2 \cos^2 r) \sin 2i + c^2 \sin 2r}.$$

The latter formula agrees with the expression found for the intensity of the extraordinary wave on the electro-magnetic theory; but Lord Rayleigh* has shown that the assumption that the densities are equal is not a legitimate one in the case of two isotropic media, since it leads to two polarising angles, and there can be little doubt that, in the case of crystalline media, the same assumption would lead to a similar result, and would therefore be one which it is not permissible to make. It thus appears that Green's theory fails to furnish a satisfactory explanation of crystalline reflection and refraction.

To work out a rigorous theory of the reflection and refraction of waves, at the surface of separation of an isotropic medium, and an anisotropic medium such as Green's, on the supposition that the velocities of propagation of the dilatational or pressural waves in both media, are very great in comparison with the velocities of propagation of the distortional waves, would be a mere question of mathematics, and could be effected without difficulty on the lines of Green's and Lord Rayleigh's investigations, when both media are isotropic. But the only physical interest of such investigations lies in their ability (or inability) to explain optical phenomena; and therefore, having regard to the failure of Green's theory to furnish satisfactory results in the case of crystalline reflection and refraction, it seems scarcely worth while to pursue such investigations.

Sir W. Thomson's Theory.

12. We shall now consider a new theory, which was proposed by Sir W. Thomson in the autumn of 1888.

When a disturbance is communicated to a homogeneous isotropic elastic medium, two waves are propagated from the centre of disturbance with different velocities, one of which is a wave of dilatation whose vibrations are perpendicular to the wave front, and whose velocity of propagation is equal to

$$(k+\tfrac{4}{3}n)^{\frac{1}{2}}/\rho^{\frac{1}{2}};$$

and the other is a distortional wave which does not involve dilatation, and whose velocity of propagation is

$$n^{\frac{1}{2}}/\rho^{\frac{1}{2}},$$

where ρ is the density, k the resistance and compression, and n the rigidity of the medium.

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* Hon. J. W. Strutt, *Phil. Mag.*, August, 1871.

In applying the theory of elastic media to explain optical phenomena, it is necessary to get rid of the difficulty which arises from the fact that such media are capable of propagating dilatational waves. This may be done by supposing that the ratio $(k+\frac{4}{3}n)^{\frac{1}{2}}/n^{\frac{1}{2}}$, of the velocity of propagation of the dilatational wave to that of the distortional wave, is either very large or very small; which requires either that k should be very large compared with n, or should be very nearly equal to $-\frac{4}{3}n$. Green adopted the former supposition, on the ground that, if the latter were true, the medium would be unstable. Sir W. Thomson, however, has recently shown that, if k is negative and numerically less than $\frac{4}{3}n$, the medium will be stable, *provided we either suppose the medium to extend all through boundless space, or give it a fixed containing vessel as a boundary.*

Putting $$U = (k+\tfrac{4}{3}n)^{\frac{1}{2}}/\rho^{\frac{1}{2}}, \quad V = n^{\frac{1}{2}}/\rho^{\frac{1}{2}},$$

it is obvious that, if a small disturbance be communicated to the medium, U will be real, provided $k+\frac{4}{3}n$ be positive, and therefore the motion will not increase indefinitely with the time, but will be periodic; but, if $k+\frac{4}{3}n$ be negative, U will be imaginary, in which case the disturbance will either increase or diminish indefinitely with the time, and the medium will either explode or collapse, and will therefore be thoroughly unstable. If $k = \frac{4}{3}n$, U will be zero, and therefore the medium will be incapable of propagating a dilatational wave. The principal difficulty in adopting this hypothesis appears to me to arise from the fact, that it requires us to suppose that the compressibility is negative,—in other words, that an increase of pressure produces an increase of volume. So far as I am aware, no medium with which we are acquainted possesses this property; and it is very difficult to form a conception of such a medium. On the other hand, it cannot be asserted that a medium possessing this property does not exist; if, therefore, we accept Sir W. Thomson's hypothesis, it follows that elastic media may be classed under the following three categories:—(i.) media which contract under pressure, for which k may have any positive value; (ii.) media which expand but do not explode or collapse under pressure, for which k may have any negative value which is numerically less than $\frac{4}{3}n$; (iii.) media which explode or collapse under pressure, for which k may have any negative value which is numerically greater than $\frac{4}{3}n$.

13. Sir W. Thomson did not apply his theory to the problem of double refraction. This has been since done by Glazebrook,* by the

* *Phil. Mag.*, Dec., 1888.

aid of a hypothesis originally suggested by Rankine and Si Stokes,* and more fully developed by Lord Rayleigh,† which c in supposing that the ether behaves like a medium which is anis as regards density, but isotropic as regards rigidity.‡

According to this hypothesis, if the axes of crystalline syn be the coordinate axes, the kinetic energy per unit of volume is

$$T = \tfrac{1}{2}\left(\rho_x \dot{u}^2 + \rho_y \dot{v}^2 + \rho_z \dot{w}^2\right),$$

and the potential energy W is of the same form as in the case isotropic medium. Putting

$$a^2 = n/\rho_x, \quad b^2 = n/\rho_y, \quad c^2 = n/\rho_z,$$

$$a'^2 - a^2 = m/\rho_x, \quad b'^2 - b^2 = m/\rho_y, \quad c'^2 - c^2 = m/\rho_z,$$

where, as usual, $\qquad m = k + \tfrac{1}{3}n,$

and using the Principle of Least Action, viz.,

$$\int dt \iiint \delta\,(T - W)\,dx\,dy\,dz = 0,$$

the equations of motion can be shown to be

$$\left.\begin{aligned} \frac{d^2u}{dt^2} &= (a'^2 - a^2)\frac{d\delta}{dx} + a^2\nabla^2 u \\ \frac{d^2v}{dt^2} &= (b'^2 - b^2)\frac{d\delta}{dy} + b^2\nabla^2 v \\ \frac{d^2w}{dt^2} &= (c'^2 - c^2)\frac{d\delta}{dz} + c^2\nabla^2 w \end{aligned}\right\} \cdots\cdots\cdots\cdots$$

and the velocity of propagation is determined by the cubic

$$\frac{l'^2}{V^2 - a^2} + \frac{m'^2}{V^2 - b^2} + \frac{n'^2}{V^2 - c^2} = \frac{k + \frac{4}{3}n}{(k + \frac{1}{3}n)\,V^2},$$

where l', m', n' are the direction cosines of the normal to the v front.

It therefore follows that, in general, there are three waves withi crystal, one of which is a quasi-dilatational wave, whilst the two are quasi-distortional waves. If $k + \frac{4}{3}n$ is absolutely zero,

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* *Brit. Assoc. Rep. on Double Refraction*, 1862.

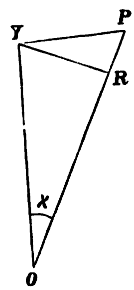
† Hon. J. W. Strutt, *Phil. Mag.*, June, 1871.

‡ [Oct. 1889.—The assumption that the density of the ether, which is a quantity, is *apparently* a function of the direction of a line, has been fully dis by Lord Rayleigh, and he has shown that it does not involve a physical im bility.]

quasi-dilatational wave disappears, leaving the two quasi-distortional or optical waves, which are propagated according to Fresnel's law.

The direction of vibration in the optical wave is not the same as in Fresnel's theory, but is found by the following construction.

Let P be the point where the ray proceeding from a point O within the crystal, meets the wave surface whose centre is O; let PY be the tangent plane to the wave surface at P, OY the perpendicular on to it from O, and draw YR perpendicular to OP. Then RY is the direction of vibration. It therefore follows that, although the direction of vibration is not the same as in Fresnel's theory, yet it lies in the plane containing the ray and the wave normal; and therefore, according to the present theory, the vibrations of a ray of polarised light on emerging from a crystal are perpendicular to the plane of polarisation.



If $POY = \chi$, the displacement S is equivalent to a displacement $S \cos \chi$ along PY, and a displacement $S \sin \chi$ along the wave normal.

Another point of importance is that, according to this theory, it is necessary to suppose that the rigidity of the ether is the same in a crystalline as in an isotropic medium; and therefore that refraction is due to a difference of density. For, if we consider two different media bounded by the plane $x = 0$, the displacements u, v, w must be continuous; also the continuity of v and w, when $x = 0$, involves the continuity of $dv/dy + dw/dz$; but, if $k + \frac{1}{3}n = 0$, the continuity of the normal stress P requires that

$$2n\left(\frac{dv}{dy} + \frac{dw}{dz}\right) = 2n'\left(\frac{dv'}{dy} + \frac{dw'}{dz}\right),$$

when $x = 0$, and this requires that $n = n'$.

14. Having given the preceding outline of the above theory, which is due to the combined efforts of Lord Rayleigh, Sir W. Thomson, and Glazebrook, we shall now consider its application to the problem of crystalline reflection and refraction.

The conditions at the surface of separation are

$$u = u_1 \qquad (26),$$

$$v = v_1 \qquad (27),$$

$$w = w_1 \qquad (28),$$

$$(m+n)\frac{du}{dx}+(m-n)\left(\frac{dv}{dy}+\frac{dw}{dz}\right)=(m'+n)\frac{du}{dx}+(m'-n)\left(\frac{dv_1}{dy}\right.$$

....

$$\frac{dv}{dx}+\frac{du}{dy}=\frac{dv_1}{dx}+\frac{du_1}{dy}\text{.....................}$$

$$\frac{du}{dz}+\frac{dw}{dx}=\frac{du_1}{dz}+\frac{dw_1}{dx}\quad\text{..................}$$

in which $m+n$ and $m'+n$ are ultimately zero. If, therefore, w garded the dilatational waves altogether, we should have six eq to determine four unknown quantities. We must therefore in a dilatational reflected and a quasi-dilatational refracted wave must be eliminated, and we shall thus obtain the correct eq for determining the amplitudes of the reflected and two re optical waves, and the deviation of the plane of polarisation former.

Let the displacements in the four optical waves be

$$\Sigma\ = S\,\epsilon^{\iota(\alpha x+\beta y+\omega t)},$$

$$\Sigma' = S'\epsilon^{\iota(-\alpha x+\beta+\omega t)},$$

$$\Sigma_1 = S_1\epsilon^{\iota(\alpha_1 x+\beta y+\omega t)},$$

$$\Sigma_2 = S_2\epsilon^{\iota(\alpha_2 x+\beta y+\omega t)};$$

and let the displacements in the dilatational reflected and dilatational refracted waves be

$$\mathfrak{S} = B\epsilon^{\iota(\gamma x+\beta y+\omega t)},$$

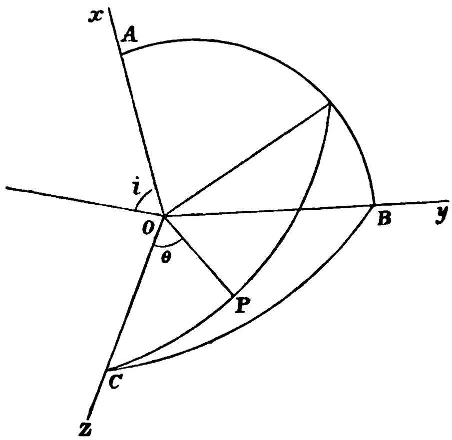
$$\mathfrak{S}_1 = B_1\epsilon^{\iota(\gamma_1 x+\beta y+\omega t)};$$

also, let I and R be the angles which the normals to these wave with the axis of x. Let θ, θ' be the angles which the directi vibrations in the incident and reflected optical waves make wi axis of z; θ_1, θ_2 the angles which the projections upon their res wave fronts of the directions of vibration in the two refracted make with this axis.

Then, adopting the notation of § 2, and omitting the co exponential factor, and also all terms involving Σ_2, which are same form as those involving Σ_1, and can therefore be supplied

end of the investigation, we have at the surface*

$$\left.\begin{aligned} u &= S\cos AP + S'\cos AP' + B\cos I \\ v &= S\cos BP + S'\cos BP' + B\sin I \\ w &= S\cos CP + S'\cos CP' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (32)$$



for the first medium. For the second medium

$$\left.\begin{aligned} u_1 &= S_1(\cos\chi_1\cos AP_1 - \sin\chi_1\cos r_1) + B_1\cos R \\ v_1 &= S_1(\cos\chi_1\cos BP_1 + \sin\chi_1\sin r_1) + B_1\sin R \\ w_1 &= S_1\cos\chi_1\cos CP_1 \end{aligned}\right\} \ldots\ldots (33).$$

Since $dw/dy = dw'/dy$ when $x = 0$, and $du/dz = du'/dz = 0$, (30) and (31) give

$$(S\cos BP - S'\cos BP')\,a + B\gamma\sin I = S_1(\cos\chi_1\cos BP_1 + \sin\chi_1\sin r_1)\,a + B_1\gamma_1\sin R \ldots\ldots\ldots\ldots (34),$$

$$(S\cos CP - S'\cos CP')\,a = S_1 a_1\cos\chi_1\cos CP_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35).$$

Since $m+n$ and $m'+n$ are ultimately zero, and $dv/dy = dv'/dy$, both

* If λ, μ, ν be the direction of vibration,

$$(V^2-a^2)\,\lambda/a^2 l = (V^2-b^2)\,\mu/b^2 m = (V^2-c^2)\,\nu/c^2 n.$$

In the case of the quasi-dilatational waves, V is zero, or at any rate very small, compared with a, b, c, whence

$$\lambda/l = \mu/m = \nu/n$$

very nearly; and therefore the direction of vibration in this wave is sensibly perpendicular to the wave front.

sides of (29) ultimately become identically equal, and this need not therefore be considered.

Now, if Λ, Λ_1 be the wave-lengths of the waves $\mathfrak{S}$, $\mathfrak{S}_1$; U velocities of propagation,

$$\gamma = \frac{2\pi}{\Lambda}\cos I, \quad \gamma_1 = \frac{2\pi}{\Lambda_1}\cos B,$$

$$\beta = \frac{2\pi}{\Lambda}\sin I = \frac{2\pi}{\Lambda_1}\sin B = -\frac{2\pi}{\lambda}\sin i = \&c.,$$

$$\omega = \frac{2\pi}{\Lambda}U = \frac{2\pi}{\Lambda_1}U_1 = \frac{2\pi V}{\lambda} = \&c.;$$

and therefore, since U, U_1 are ultimately zero, Λ, Λ_1 are also ul zero; whence $I = 0$, $B = 0$, and therefore γ, γ_1 are ultimately

Also, $$\gamma \sin I = \frac{2\pi}{\Lambda}\cos I \sin I = \beta \cos I.$$

Writing out the equation $u = u_1$ in full, multiplying by subtracting from (34), we obtain

$$(S\cos BP - S'\cos BP')\,\alpha - (S\cos AP + S'\cos AP')\,\beta$$
$$= S_1(\cos\chi_1\cos BP_1 + \sin\chi_1\sin r_1)\,\alpha_1 - S_1(\cos\chi_1\cos AP_1 - \sin\chi_1$$
....

From the preceding investigation we see that B and B_1 are n but finite, and therefore the existence of the waves $\mathfrak{S}$, $\mathfrak{S}_1$ ca entirely ignored; but, since $I = B = 0$ the terms involving disappear from the equation $v = v'$, which gives

$$S\cos BP + S'\cos BP' = S_1(\cos\chi_1\cos BP_1 + \sin\chi_1\sin r_1)....$$

and the equation $w = w'$ gives

$$S\cos CP + S'\cos CP' = S_1\cos\chi_1\cos CP_1 \text{ ....}$$

Equations (35), (36), (37), and (38) contain the complete s of the problem.

Now

$$\cos AP = \sin i\sin\theta, \quad \cos BP = \cos i\sin\theta, \quad \cos CP = \cos$$

with similar expressions for $\cos AP_1$, &c.; also,

$$\cos AP' = \sin i\sin\theta', \quad \cos BP' = -\cos i\sin\theta', \quad \cos CP' = \text{co}$$

whence

$$\left.\begin{aligned}(S\cos\theta - S'\cos\theta')\cot i &= S_1\cot r_1\cos\chi_1\cos\theta_1\\(S\sin\theta + S'\sin\theta')\operatorname{cosec} i &= S_1\operatorname{cosec} r_1\cos\chi_1\sin\theta_1\\(S\sin\theta - S'\sin\theta')\cos i &= S_1(\cos r_1\cos\chi_1\sin\theta_1 + \sin r_1\sin\chi_1)\\S\cos\theta + S'\cos\theta &= S_1\cos\chi_1\cos\theta_1\end{aligned}\right\}.$$

in which equations we are to recollect, that we are to add to the right-hand sides terms in S_2 similar to those involving S_1.

15. The preceding equations may also be obtained by a process which does not involve the introduction of the dilatational waves.

Since the continuity of u, v, w involves the continuity of their differential coefficients with respect to y and z, (26), (30) and (31) involve the continuity of the rotations η and ζ; also, since $m = m' = -n$, both sides of (29) are identically equal, and therefore this equation disappears; we are thus left with (27) and (28). The surface conditions are therefore

$$v = v_1, \quad w = w_1;$$

$$\eta = \eta_1, \quad \zeta = \zeta_1;$$

which furnish four equations to determine the four unknown quantities.

Equations (39) determine the *amplitudes* of the reflected and refracted light, but, according to the electro-magnetic theory, the intensity is to be measured by the *energy*; and, in order to be consistent, the intensity, according to the present theory, ought to be measured in the same way.

If λ, μ, ν be the direction of displacement, the kinetic energy per unit of volume corresponding to the displacement Σ_1 is

$$\frac{1}{2}\, n \left(\frac{\lambda^2}{a^2} + \frac{\mu^2}{b^2} + \frac{\nu^2}{c^2}\right) S_1^2\, \omega^2 \cos^2(a_1 x + \beta y + \omega t);$$

and therefore the intensity is proportional to

$$\left(\frac{\lambda^2}{a^2} + \frac{\mu^2}{b} + \frac{\nu^2}{c^2}\right) S_1^2.$$

This quantity* can easily be shown to be equal to $(S_1/V_1)^2 \cos^2 \chi_1$, and therefore, if A, A', A_1, A_2 be the square roots of the intensities, we shall have

$$\frac{S}{A \sin i} = \frac{S'}{A' \sin i} = \frac{S_1 \cos \chi_1}{A_1 \sin r_1} = \frac{S_2 \cos \chi_2}{A_2 \sin r_2}.$$

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* The direction cosines of PY are proportional to λ/a^2, μ/b^2, ν/c^2; whence

$$\cos \chi = \frac{\lambda^2/a^2 + \mu^2/b^2 + \nu^2/c^2}{(\lambda^2/a^4 + \mu^2/b^4 + \nu^2/c^4)^{\frac{1}{2}}};$$

but

$$\cos \chi = \frac{V}{r} = \frac{1}{r} \left(\frac{\lambda^2/a^2 + \mu^2/b^2 + \nu^2/c^2}{\lambda^2/a^4 + \mu^2/b^4 + \nu^2/c^4}\right)^{\frac{1}{2}},$$

whence

$$\frac{1}{r^2} = \frac{\lambda^2}{a^2} + \frac{\mu^2}{b^2} + \frac{\nu^2}{c^2} = \frac{\cos^2 \chi}{V^2}.$$

Substituting in (39), it is at once seen that the equations for determining the intensities are the same as those deduced by means of the electro-magnetic theory.

The principal results of the theory are—(i.) that the equations which solve the problem of crystalline reflection and refraction, are the same as those which are obtained by means of the electro-magnetic theory, and which have experimentally been proved to be fairly, although not accurately, true; (ii.) that the theory leads to a wave-surface which is approximately, although not accurately, Fresnel's wave-surface, unless k is absolutely and not approximately equal to $-\frac{4}{3}n$. Also, as soon as the assumptions have been made that k is equal or nearly so to $-\frac{4}{3}n$, and that double refraction arises from the circumstance, that crystalline media behave as if they were anisotropic as regards density; results which can be proved to be approximately true, are capable of being deduced without the aid of any of those additional assumptions, which in many physical problems are indispensable in order to obtain a particular analytical result.

On the Uniform Deformation in Two Dimensions of a Cylindrical Shell of Finite Thickness, with application to the General Theory of Deformation of Thin Shells. By LORD RAYLEIGH, Sec. R.S.

[*Read June 13th*, 1889.]

The theory of a thin uniform shell of elastic isotropic material, slightly deformed from an originally curved condition, does not seem to be yet upon an entirely satisfactory footing. If the middle surface be extended, it is clear* that, to a first approximation, the potential energy per unit of area is

$$2nh\left\{\sigma_1^2+\sigma_2^2+\tfrac{1}{2}\varpi^2+\frac{m-n}{m+n}(\sigma_1+\sigma_2)^2\right\} \quad \ldots\ldots\ldots\ldots\ldots(1),$$

where $2h$ denotes the thickness of the shell; m, n the elastic constants

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* See Lamb, *Proc. Math. Soc.*, Dec. 1882. Also *Proc. Roy. Soc.*, XLV. (1888), p. 111, equation (13).

of Thomson and Tait's notation; σ_1, σ_2, ϖ the elongations and shear of the middle surface at the place under consideration. Again, if the deformation be such that the middle surface remain unextended, so that (1) vanishes, it is tolerably clear that the potential energy takes the form

$$\tfrac{2}{3}nh^3\left\{\left(\delta\frac{1}{\rho_1}\right)^2+\left(\delta\frac{1}{\rho_2}\right)^2+\frac{m-n}{m+n}\left(\delta\frac{1}{\rho_1}+\delta\frac{1}{\rho_2}\right)^2+2\tau^2\right\} \quad\ldots\ldots(2),$$

where $\delta\rho_1^{-1}$, $\delta\rho_2^{-1}$ are the changes of principal curvatures of the middle surface, and τ is determined by the angle (χ) through which the principal planes are shifted according to the equation

$$\tau = 2\chi\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).*$$

But when the middle surface undergoes stretching, so that (1) is finite, while yet the circumstances of the problem forbid us to remain satisfied with terms involving the first power of h, it is a more difficult question to determine the expression for the potential energy complete to the order h^3. An investigation of this problem has, however, been given by Mr. Love, and his result† is exhibited in terms of σ_1, σ_2, ϖ, and of quantities depending upon these, and upon the alterations of curvature of the middle surface.

It may, indeed, be an under-statement of the case to speak of the problem as difficult, for to all appearance it may well be impossible in the form proposed. When the middle surface is plane, or when, though originally curved, it remains unstretched, there is no difficulty in supposing that the faces are exempt from imposed force. But when the middle surface of a shell is originally curved, and undergoes extension, equilibrium cannot be maintained without the cooperation of forces normal to the shell, and acting either upon the interior or upon the faces. It is easy to understand that the precise seat of these forces may be a matter of indifference, so far as the term of the first order (1) is concerned; but is there any reason for anticipating that there would be no effect upon the term of the third order? Rather, it would appear probable that there is no expression for the potential energy complete to the order h^3, in the absence of more definite suppositions as to the manner of application of the normal forces necessary in the general case. These doubts led me

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* See Love, *Phil. Trans.* CLXXIX. (1888), A, pp. 505, 512; Rayleigh, *loc. cit.*, p. 113.

† *Loc. cit.*, p. 505.

to think an investigation desirable, which should be based upon the general equations of elasticity, and conducted without the aid of approximations of ill-defined significance. For this purpose I have chosen the simplest problem involving the questions at issue, that namely of the deformation in two dimensions of a shell originally cylindrical.

Taking polar coordinates, let u, v* be the displacements at the point (r, θ) parallel to r and θ respectively. The displacement w, parallel to the axis of the cylinder, vanishes by hypothesis. The strains relative to these directions are†

$$e = \frac{du}{dr}, \quad f = \frac{d}{d\theta}\left(\frac{v}{r}\right) + \frac{u}{r}, \quad g = 0 \quad \text{..............(4)},$$

$$a = 0, \quad b = 0, \quad c = r\frac{d}{dr}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{du}{d\theta} \quad \text{..............(5)}.$$

The stresses P, Q, R, S, T, U corresponding to these strains are given by

$$P = (m+n)\,e + (m-n)\,f, \quad Q = (m+n)\,f + (m-n)\,e \text{......(6)},$$

$$S = 0, \quad T = 0, \quad U = nc \quad \text{.....................(7)}.$$

If there be no internal impressed forces, the equations of equilibrium are

$$\frac{d}{dr}(Pr) + \frac{dU}{d\theta} - Q = 0 \quad \text{.......................(8)},$$

$$\frac{d}{dr}(Ur^2) + r\frac{dQ}{d\theta} = 0 \text{..........................(9)}.$$

We will now limit the problem by the supposition that the strains and stresses are independent of θ. Thus

$$dU/d\theta = 0, \quad dQ/d\theta = 0 \text{ ....(10)};$$

and (8), (9) reduce to

$$\frac{d}{dr}(Pr) - Q = 0 \text{............................(11)},$$

$$\frac{d}{dr}(Ur^2) = 0 \text{ ....(12)}.$$

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* This notation differs from that employed in my former papers, where u denoted the displacement parallel to the axis.

† Ibbetson's "Elastic Solids," 1887, p. 238.

From (12) it follows that Ur^3 is an absolute constant. Hence if, as we will now suppose, U vanishes over the cylindrical faces of the shell, it necessarily vanishes throughout the interior. Thus, by (7),

$$c = 0 \quad \text{.................................} (13)$$

throughout. From (5) and (13),

$$\frac{d}{dr}\left\{r^3 \frac{d}{dr}\left(\frac{v}{r}\right)\right\} = -\frac{d}{d\theta}\frac{du}{dr} = -\frac{de}{d\theta} = 0,$$

by hypothesis. Hence

$$v = C_1 + C_2 r \quad \text{..............................} (14),$$

where C_1, C_2 are independent of r, but may be functions of θ. Again, from (5) and (14),

$$\frac{du}{d\theta} = -r^3 \frac{d}{dr}\left(\frac{v}{r}\right) = C_1;$$

so that, by (4),

$$\frac{df}{d\theta} = \frac{1}{r}\frac{d^2C_1}{d\theta^2} + \frac{d^2C_2}{d\theta^2} + \frac{C_1}{r}.$$

But $df/d\theta = 0$, by supposition. Accordingly,

$$\frac{d^2C_1}{d\theta^2} + C_1 = 0, \qquad \frac{d^2C_2}{d\theta^2} = 0;$$

or

$$C_1 = H\cos\theta + K\sin\theta, \quad C_2 = C + D\theta \text{...............} (15),$$

where H, K, C, D are absolute constants. Thus, by (14),

$$v = H\cos\theta + K\sin\theta + (C + D\theta)\,r \text{..................} (16),$$

and

$$u = H\sin\theta - K\cos\theta + \phi(r) \text{ ....} (17),$$

where $\phi(r)$ is a function of r which is, so far, arbitrary. Again, by (4),

$$e = \phi'(r), \quad f = D + r^{-1}\phi(r) \text{ ....} (18),$$

indicating that the strains are independent of the coefficients H, K, C. The terms in H, K represent merely a displacement of the cylinder without rotation or strain, and the term in C represents simple rotation of the cylinder about its axis as a rigid body. They may be omitted without loss of anything material to the present inquiry.

So far, we have made no use of the condition (11) that there is no internal force in the radial direction. It is by means of this that the

form of ϕ must be determined. From (6), (18),

$$P = (m+n)\,\phi'(r) + (m-n)\,\{D + r^{-1}\phi(r)\} \quad\text{.........(19)};$$

$$Q = (m+n)\,\{D + r^{-1}\phi(r)\} + (m-n)\,\phi'(r) \quad\text{.........(20)};$$

so that, by (11),

$$r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} - \phi = \frac{2nDr}{m+n} \quad\text{.....................(21)},$$

the differential equation which must be satisfied by ϕ.

The solution of (21) is

$$\phi = Ar + Br^{-1} + \frac{nD}{m+n}\,r\log r \text{.....................(22)},$$

where A and B are arbitrary constants. Corresponding to (22),

$$e = A - Br^{-2} + \frac{nD}{m+n}(\log r + 1) \text{..................(23)},$$

$$f = D + A + Br^{-2} + \frac{nD}{m+n}\log r \quad\text{..................(24)};$$

and from (16), (17), if $H = K = C = 0$,

$$u = Ar + Br^{-1} + \frac{nD}{m+n}\,r\log r \quad\text{..................(25)},$$

$$v = Dr\theta \text{...(26)}.$$

We have now to consider the potential energy of strain. The general expression for the energy per unit of volume in a strained solid is

$$W = \tfrac{1}{2}(m+n)(e^2+f^2+g^2) + (m-n)(fg+ge+ef) + \tfrac{1}{2}n(a^2+b^2+c^2)$$
$$\text{...............(27)}.$$

By (4), (5), (13), we have

$$a = 0, \quad b = 0, \quad c = 0, \quad g = 0;$$

so that (27) reduces to

$$W = \tfrac{1}{2}(m+n)(e^2+f^2) + (m-n)\,ef = \tfrac{1}{2}m\,(e+f)^2 + \tfrac{1}{2}n\,(e-f)^2 \text{...}\ (28).$$

In the present problem

$$e+f = D + 2A + \frac{nD}{m+n}(2\log r + 1) \quad\text{..............(29)},$$

$$e-f = -D - 2Br^{-2} + \frac{nD}{m+n} \quad\text{........................(30)}.$$

Before proceeding further, we will consider in detail the very simple case which arises when $D = 0$. We have

$$e = A - Br^{-2}, \quad f = A + Br^{-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (31);$$

$$u = Ar + Br^{-1}, \quad v = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (32).$$

These equations constitute the solution of the problem of the deformation of a complete cylindrical shell (of finite thickness) under the action of hydrostatic pressures (or tractions) upon its inner and outer faces.* For the radial stress at any point, we have

$$P = 2mA - 2nBr^{-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33).$$

Thus, if the stress upon the inner face $r = r_1$ be Π_1, and upon the outer face $r = r_2$ be Π_2,

$$\left.\begin{aligned} \Pi_1 &= 2mA - 2nBr_1^{-2} \\ \Pi_2 &= 2mA - 2nBr_2^{-2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (34),$$

by which A and B are determined.

The expression for the energy becomes, by (28), (29), (30),

$$W = 2mA^2 + 2nB^2r^{-4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35).$$

The whole potential energy per unit of length parallel to the axis is given by

$$2\pi \int_{r_1}^{r_2} Wr\,dr = 2\pi \left\{ mA^2 (r_2^2 - r_1^2) - nB^2 (r_2^{-2} - r_1^{-2}) \right\} \ldots\ldots (36).$$

In order to apply this result to a thin shell, we will write

$$r_1 = a - h, \quad r_2 = a + h,$$

where $2h$ denotes the thickness of the shell, and a the radius of the middle surface. Thus

$$\int Wr\,dr = 4ah \left\{ mA^2 + n \frac{B^2}{r_1^2 r_2^2} \right\}$$

$$= 4ah \left\{ mA^2 + nB^2a^{-4} + 2nB^2a^{-6}h^2 \right\} \ldots\ldots\ldots\ldots\ldots\ldots (37),$$

approximately.

The extension of the middle surface is here, by (31),

$$\sigma = A + Ba^{-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38).$$

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* Ibbetson, *loc. cit.*, pp. 313, 314.

Since there are two independent variables A, B, or Π_1, Π_2, in (37), it is clear that the potential energy cannot, in strictness, be determined by σ only. Let us, however, inquire to what order of approximation the energy is a function of σ, when h is regarded as small.

If ϖ denote the ratio of surface forces by which the deformation is maintained, we have, from (34),

$$mA(1-\varpi) = nB(r_2^{-2} - \varpi r_1^{-2});$$

from which, and (38),

$$\sigma = A\left\{1 + \frac{m}{n}\frac{(1-\varpi)a^{-2}}{r_2^{-2} - \varpi r_1^{-2}}\right\}$$

$$= Ba^{-2}\left\{1 + \frac{n}{m}\frac{r_2^{-2} - \varpi r_1^{-2}}{(1-\varpi)a^{-2}}\right\} \quad \text{(39)},$$

equations giving A and B in terms of σ and ϖ. Using these, we find, on reduction,

$$mA^2 + \frac{nB^2}{r_1^2 r_2^2} = \frac{mn\sigma^2}{m+n}\left\{1 + \frac{2m}{m+n}\frac{h^2}{a^2} + \frac{4mn}{(m+n)^2}\frac{h^2}{a^2}\left(\frac{1+\varpi}{1-\varpi}\right)^2\right\},$$

the term containing the first power of h disappearing. Thus, for the potential energy per unit of area of the shell, we obtain

$$a^{-1}\int Wr\,dr = \frac{4mn\sigma^2 h}{m+n}\left\{1 + \frac{2m}{m+n}\frac{h^2}{a^2} + \frac{4mn}{(m+n)^2}\frac{h^2}{a^2}\left(\frac{1+\varpi}{1-\varpi}\right)^2\right\} \quad \text{....(40)}.$$

The term in h agrees, as might have been expected, with (1).* But, when the approximation is carried so far as to include h^2, (40) depends upon ϖ as well as upon σ. If the normal forces are limited to one surface, $\varpi = 0$, or $\varpi = \infty$. In either case

$$(1-\varpi)^2/(1+\varpi)^2 = 1,$$

and

$$a^{-1}\int Wr\,dr = \frac{4mn\sigma^2 h}{m+n}\left\{1 + \frac{2m}{m+n}\frac{h^2}{a^2} + \frac{4mn}{(m+n)^2}\frac{h^2}{a^2}\right\} \quad \text{.........(41)}.$$

The energy involved in a given extension of the middle surface is thus the same, whether the necessary normal force be an internal pressure or an external traction; but the case is otherwise if the forces be distributed. When the work is equally divided between the two

* ϖ has there a different meaning from that belonging to it in (40). In (1) $\varpi = 0$, $\sigma_2 = 0$, for the purposes of the present problem.

surfaces, so that there is (for example) a pressure upon the internal surface and a traction upon the external surface, $\varpi = -1$; and

$$a^{-1}\int Wr\,dr = \frac{4mn\sigma^2 h}{m+n}\left\{1+\frac{2m}{m+n}\frac{h^2}{a^2}\right\} \quad\text{..............(42).}$$

It will be seen that, in order to give rise to this discrepancy, it is not necessary to suppose the introduction of surface forces more powerful than are actually required to maintain the deformation. This instance is sufficient to show that the potential energy of deformation cannot, in general, be expressed in terms of extensions and changes of curvature of the middle surface, when it is necessary to include terms of order h^3, without further information as to the manner in which the surface forces are applied. According to Mr. Love's results,* the expression for the energy in the present problem should reduce to its first term; whereas (40) indicates that there is no manner of application of the surface forces by which such a result could be brought about.

We will now abandon the restriction to $D = 0$. It will then be possible to find a deformation such that, not only is there no impressed force upon the interior of the shell, but also none upon either of the surfaces. Under these circumstances the stresses between contiguous parts must reduce themselves to a simple couple.

From (6), (29), (30), we find

$$P = m(e+f)+n(e-f)$$

$$= 2mA-2nBr^{-2}+D\left\{m+\frac{2mn\log r}{m+n}\right\} \quad\text{..............(43).}$$

If $P = 0$, both when $r = r_1$ and when $r = r_2$, the values of A, B, in terms of D, are

$$A = -D\left\{\frac{1}{2}+\frac{n}{m+n}\frac{r_2^{-2}\log r_1 - r_1^{-2}\log r_2}{r_2^{-2}-r_1^{-2}}\right\} \quad\text{.......... (44),}$$

$$B = \frac{mD}{m+n}\frac{\log(r_2/r_1)}{r_2^{-2}-r_1^{-2}} \quad\text{.................................... (45).}$$

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* *Loc. cit.*, equations (12), (18). [December, 1889. I have been reminded by the Secretary that in the investigation of Mr. Love it is expressly supposed (p. 504) that no surface tractions are applied. But the absence of normal forces would, as it appears to me, be equivalent to a limitation upon the generality of the middle surface extensions, σ_1, σ_2.]

These values, substituted in (23), (24), (25), (26), determine a definite type of deformation, satisfying the conditions that there shall be no internal or surface forces, and that the strains shall be independent of θ, and this without any supposition limiting the thickness of the shell.

From the expression for Q in terms of e and f, or, more readily, by means of (11), we may verify that

$$\int_{r_1}^{r_2} Q\,dr = 0 \quad\text{..............................(46).}$$

In order to apply these results to a thin shell, we write, as before,

$$r_1 = a-h, \quad r_2 = a+h;$$

thus

$$A = -D\left\{\frac{1}{2}+\frac{n}{m+n}\left(\log a+\frac{1}{2}+\frac{h^2}{6a^2}\right)\right\} \quad\text{............ (47),}$$

$$B = -\frac{mD}{m+n}\,\frac{a^2}{2}\left(1-\frac{5h^2}{3a^2}\right) \quad\text{..............................(48).}$$

Corresponding to these, from (29), (30),

$$e+f = \frac{nD}{m+n}\left\{2\log\frac{r}{a}-\frac{h^2}{3a^2}\right\} \quad\text{........................ (49),}$$

$$e-f = -\frac{mD}{m+n}\left\{1-\frac{a^2}{r^2}\left(1-\frac{5h^2}{3a^2}\right)\right\} \quad\text{..................(50);}$$

or, if $r = a+\rho$,

$$e+f = \frac{nD}{m+n}\left\{\frac{2\rho}{a}-\frac{\rho^2}{a^2}-\frac{h^2}{3a^2}\right\} \quad\text{.................. (51),}$$

$$e-f = \frac{-mD}{m+n}\left\{\frac{2\rho}{a}-\frac{3\rho^2}{a^2}+\frac{5h^2}{3a^2}\right\} \quad\text{.................. (52).}$$

The strains e, f both vanish approximately when $r = a$. By (6),

$$P = \frac{2mnD}{m+n}\left\{\frac{\rho^2}{a^2}-\frac{h^2}{a^2}\right\} \quad\text{.......................... (53),}$$

$$Q = \frac{4mnD}{m+n}\left\{\frac{\rho}{a}-\frac{\rho^2}{a^2}+\frac{h^2}{3a^2}\right\} \quad\text{.................. (54).}$$

We will now calculate the potential energy of deformation. From (28), (51), (52),

$$W = \frac{mn^2D}{(m+n)^2}\left\{\frac{2\rho^2}{a^2}-\frac{2\rho^3}{a^3}-\frac{2\rho h^2}{3a^3}\right\}+\frac{m^2nD}{(m+n)^2}\left\{\frac{2\rho^2}{a^2}-\frac{6\rho^3}{a^3}+\frac{10\rho h^2}{3a^2}\right\};$$

so that, for the potential energy per unit of area, we get

$$a^{-1}\int W r\,dr = \int_{-h}^{+h} W\left(1+\frac{\rho}{a}\right)d\rho = \frac{4mnD^2}{3\,(m+n)}\,\frac{h^3}{a^2}\,\ldots\ldots\ldots(55),$$

the next term involving h^5.

In order to connect this with the change of curvature of the middle surface, we require the expression for u. From (25),

$$u = -Dr\left\{\frac{1}{2}+\frac{n}{m+n}\left(\log a+\frac{1}{2}+\frac{h^2}{6a^2}\right)\right\}$$
$$-\frac{mDr^{-1}a^2}{2\,(m+n)}\left(1-\frac{5h^2}{3a^2}\right)+\frac{nD}{m+n}\,r\log r\,\ldots\ldots\ldots\ldots\ldots(56);$$

so that the value of u at the middle surface ($r = a$) is, approximately,

$$u = -aD\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(57).$$

Now $a+u$ is the radius of curvature of the middle surface after deformation, or $\delta\rho_1 = u$. Thus

$$\left(\delta\,\frac{1}{\rho_1}\right)^2 = \frac{u^2}{a^4} = \frac{D^2}{a^2}.$$

The expression for the energy per unit area of surface is thus

$$a^{-1}\int W r\,dr = \frac{4mnh^3}{3\,(m+n)}\left(\delta\,\frac{1}{\rho_1}\right)^2,$$

in agreement with (2); for in the present application

$$\delta\,\frac{1}{\rho_2} = 0,\quad \tau = 0.$$

It is evident that the rigorous solution from which we started is available for continuing the approximation, should it be thought desirable to retain higher powers of h.

The solution of the problem of the bending of a cylindrical shell, here put forward, favours then the idea that it is only when the middle surface of a curved shell remains unextended that it is possible to express the potential energy to the order h^3 in terms merely of the extensions and curvatures of the middle surface.

On the Figures of a certain Class of Cubic Curves and th *Concomitants.* *By* J. J. WALKER.

[*Read June 13th*, 1889.]

When recently occupied with the drawing of figures to illustr the properties of the Poloid of a line relative to a cubic, I was led consider the desirability of an attempt being made to trace the figur of the Hessian, Cayleyan, and Quippian of some non-singular cub I decided to select an example from that class which is the Hessi of its own Hessian, as being always reducible to a trinomial for and always non-singular, when a proper cubic; and, for ma reasons, deserving of special study. Before explaining the tracing the curves for special numerical values of the coefficients, it may desirable to consider the general equations to this class of cubics, a its principal concomitants. In a paper which will be found in t *Quarterly Journal*, XVI., pp. 188, 189, Mr. W. J. C. Sharp has point out that the equation of every plane cubic for which the sextic i variant vanishes may be reduced to the form

$$u \equiv ax^3 + 3by^2z + 3cz^2x^* = 0 \quad \dots\dots\dots\dots\dots\dots (1$$

its Hessian being of the same form, except that x and z are inte changed, viz.,

$$w \equiv abcx^2z - ab^2y^2x - bc^2z^3 = 0 \quad \dots\dots\dots\dots\dots\dots (2$$

while the concomitants P and Q are of the same forms in line coo dinates as u and w respectively are in point coordinates, viz.,

$$P = b^2c\alpha^3 + 2abc\beta^2\gamma + ab^2\gamma^2\alpha \dots\dots\dots\dots\dots\dots (3$$

$$Q = 2\,(a^2b^3\gamma^3 + 6ab^2c^2\beta^2\alpha - 3ab^3c\alpha^2\gamma) \dots\dots\dots\dots\dots\dots (4$$

The reciprocal of u may be most conveniently arranged by powe of β; it is

$$v \equiv 4ac^3\beta^6 + 24abc^2\gamma\alpha\beta^4 + 3\,(-b^2c^2\alpha^4 + 10ab^2c\gamma^2\alpha^2 + 3a^2b^2\gamma^4)\,\beta^2$$
$$-4b^3\,(3c\alpha^2 + a\gamma^2)\,\alpha^3\gamma \dots\dots\dots\dots\dots\dots (5$$

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* As to signs: a being regarded as essentially positive, by choosing the sign z, the term by^2z may be made positive, so that the term cz^2x shall be alone of doubt sign. Or, b may be considered positive, and by choosing the sign of x the te cz^2x may be made positive, so that a shall be the only coefficient of doubtful sig The latter is the convention assumed in what follows.

and it is convenient to remark that, if

$$\beta'^2 \equiv c\beta^2 + 2b\gamma\alpha,$$

then $$cv \equiv 4ca\beta'^6 + 3b^2(-c^2\alpha^4 - 6ca\gamma^2\alpha^2 + 3ab\gamma^4)\beta'^2$$
$$-6(c^2\alpha^4 + 3a^2\gamma^4)\,b^3\gamma\alpha \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

The equation to P in point coordinates may, plainly, be deduced from v by writing x for α, z for γ, $3b^2c$ for a, $2abc$ for b, and ab^2 for c; viz., it is (the factors $12a^2b^5c$ rejected)

$$P = ab^3y^6 + 12ab^2czxy^4 + (27bc^3z^4 + 30abc^2z^2x^2 - a^2bcx^4)\,y^2$$
$$-8ac^2x^3z\,(ax^2 + cz^2) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7);$$

or, writing, $$y'^2 = by^2 + 4czx,$$

$$P = ay'^6 + c\,(27c^2z^4 - 18caz^2x^2 - a^2x^4)\,y'^2 - 4c^2zx\,(a^2x^4 + 27c^2z^4) = 0 \;\ldots(8).$$

Similarly, the equation to Q, in point coordinates, may be deduced from the reciprocal of u by writing z for α, x for γ, y for β, ab for a, $2c^2$ for b, $-bc$ for c; which changes give, the factors $-4a^3b$ being dropped,

$$Q = ab^3y^6 - 12ab^2czxy^4 + 3\,(bc^3z^4 + 10abc^2z^2x^2 - 3a^2bcx^4)\,y^2$$
$$+8c^3z^3x\,(ax^2 - 3cz^2) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

If $$y'^2 = a\,(by^2 - 4czx),$$

the equation to Q may be written

$$Q = y'^6 + 3ac\,(a^2z^4 - 6caz^2x^2 - 3a^2x^4)\,y'^2 - 12a^2c^2xz\,(3a^2x^4 + c^2z^4) = 0 \;\ldots\ldots\ldots\ldots(10).$$

The satellite line of $$\alpha x + \beta y + \gamma z = 0$$

with respect to u, calculated from the formula (49), *Phil. Trans.*, 1888, A, p. 170, is

$$4ab^2\,(3bc\alpha^3\gamma + 3c^2\alpha^2\beta^2 + ab\alpha\gamma^3 - 3ac\beta^2\gamma^2)\,x - 8ab^3\,(3c\alpha^2 + a\gamma^2)\,\beta\gamma y$$
$$+ (9b^3c^2\alpha^4 - 6ab^3c\alpha^2\gamma^2 - 24ab^2c^2\alpha\beta^2\gamma - 12abc^3\beta^4 + a^2b^3\gamma^4)\,z = 0 \ldots(11).$$

Multiplying this by $(\alpha x + \beta y + \gamma z)^2$, and adding to it the product (1) (5), the equation of the three tangents to u, at the points

common to it and $\alpha x+\beta y+\gamma z$, is found to be

$$\begin{aligned}&\{9ab^2(c\alpha^2+a\gamma^2)^2+4a^2c^2(c\beta^2+6b\gamma\alpha)\beta^2\}\beta^2x^3-8ab^3(3c\alpha^2+a\gamma^2)\beta^3\gamma y^3\\&+b\{b^2(3c\alpha^2-a\gamma^2)^2-12ac^2(c\beta^2+2b\gamma\alpha)\beta^2\}\gamma^2z^3\\&+24ab^2c(c\alpha^2-a\gamma^2)\alpha\beta^3x^2y\\&+3b\{3b^2\alpha^2(c\alpha^2+a\gamma^2)^2-4ac\alpha\beta^2(c^2\alpha\beta^2+2ab\gamma^3)\}x^2z\\&+12ab^2\beta^2\{c\alpha^2(c\beta^2-3ba\gamma)-a\gamma^2(c\beta^2+ba\gamma)\}y^2x\\&+12b^2\gamma\{-b^2\alpha^3(3c\alpha^2+a\gamma^2)+3abc\alpha^2\beta^2\gamma+4ac^2\alpha\beta^4+a^2b\beta^2\gamma^3\}y^2z\quad\ldots(12).\end{aligned}$$

These are the only equations used in the sequel.

To trace u and w, which, plainly, are both non-singular: eliminating y^2 between (1) and (2),

$$(3\pm2\sqrt{3})\,ax^2=3cz^2,$$

or if, for shortness,

$$p=\sqrt{3}+1,\quad q=\sqrt{3}-1,\quad r=\sqrt{2\sqrt{3}},$$

p, q, r being positive arithmetical magnitudes, then the real values of x/z are determined by

$$p^2ax^2=r^2cz^2,$$

or

$$q^2ax^2=-r^2cz^2,$$

according as a, c are of like or unlike sign; and, correspondingly, by substitution in either u or w,

$$\left.\begin{aligned}3by^2&=pr\sqrt{ca}\,x^2\\ \text{with}\quad p\sqrt{a}\,x&=-r\sqrt{c}\,z\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i.}),$$

$$\left.\begin{aligned}\text{or}\quad 3by^2&=qr\sqrt{-ca}\,x^2\\ \text{with}\quad q\sqrt{a}\,x&=r\sqrt{-c}\,z\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii.}),$$

gives, in the several cases, the coordinates of the points of real inflexion of u and w, in the same right line with the inflexion-point

$$x=0,\quad z=0.$$

The tangents to w at the points (i.) are, i[illegible] ...ss

x' is written for $\sqrt{a}\,x$ [illegible]

y' „ „ $\sqrt{b\sqrt{a}}$ [illegible] ....

z' „ „ $\sqrt{c}\,z$ [illegible]

taking the absolute numerical value of a, supposed to be the only one of the coefficients of doubtful sign,

$$2p^{\frac{3}{2}}x' \pm 2r^{\frac{3}{2}}y' + p^{\frac{1}{2}}r^{3}z' = 0 \quad \text{......................(iv.)},$$

the pair of tangents at the points (ii.) being

$$2q^{\frac{3}{2}}x' \pm 2r^{\frac{3}{2}}y' + q^{\frac{1}{2}}r^{3}z' = 0 \quad \text{........................(v.)}.$$

Eliminating y' between these equations successively, and $u = 0$,

$$r^3x'^3 + 3p^3x'^2z' + 3\,(1+p^2)\,r^3x'z'^2 + 9pr^2z'^3 = 0,$$

or

$$(px'+rz')(rx'+3pz')^2 = 0 \quad \text{......................(vi.)};$$

viz., the tangents to w at the points (i.) touch u at points on

$$rx'+3pz' = 0 \quad \text{..............................(vii.)},$$

and, similarly, the tangents to w at the points (ii.) touch u on

$$rx'-3qz' = 0 \quad \text{...........................(viii.)}.$$

Again, the real tangents to u, at the same points of inflexion, are

$$prx' \mp p^{\frac{3}{2}}r^{\frac{1}{2}}y' - 2z' = 0 \quad \text{........................(ix.)},$$

if a, c are of the same sign;

$$qrx' \mp q^{\frac{3}{2}}r^{\frac{1}{2}}y' - 2z' = 0 \quad \text{........................(x.)},$$

if of contrary signs. Eliminating y between (ix.) and (2),

$$p\,(px'+rz')(rx'-pz')^2 = 0;$$

viz., the tangents to u at the points (i.) touch w on the line

$$rx'-pz' \quad \text{or} \quad r\sqrt{a}\,x - p\sqrt{c}\,z = 0 \quad \text{..................(xi.)}.$$

Similarly, those at the points (ii.) touch w on the line

$$rx'-qz' \quad \text{or} \quad r\sqrt{-a}\,x - q\sqrt{c}\,z = 0 \quad \text{...............(xii.)}.$$

To obtain a general conception of the form of u, let

$$z = kx,$$

giving (1)

$$3kby^2 = -(a+3ck^2)\,x^2.$$

Supposing a to be essentially positive, as well as b and c, y/x will only be real for negative values of k. In this case, then, the curve u consists of a single real part only, viz., a serpentine branch with three real inflexions lying within one of the angles contained by the tangents to it and its Hessian at any one of the three real points of inflexion, and having one real asymptote. Or, it consists of two

infinite branches having one common asymptote, and each one separate real asymptote.

But, supposing a to be essentially negative, say

$$a = -d,$$

then
$$3kby^2 = (d-3ck^2)\,x^2,$$

and y/x will be real for all positive values of k not exceeding $\sqrt{d/3c}$, and for negative values of k exceeding that value,

$$3cz^2 - dx^2 = 0$$

being tangents to u.

The axis of inflexions, (ii.), p. 384,

$$q\sqrt{d}\,x = r\sqrt{c}\,z,$$

plainly lies between $z = 0$ and the tangent

$$\sqrt{d}\,x = \sqrt{3c}\,z;$$

since
$$q/r = (\sqrt{3}-1)/\sqrt{2\sqrt{3}},$$
$$< (\sqrt{3}-1)/\sqrt{3},$$
$$< 1/\sqrt{3}.$$

In this case, then, the serpentine branch with three real inflexions lies between the inflexional tangent $z = 0$ and the ordinary tangent $\sqrt{d}\,x = \sqrt{3c}\,z$, drawn from the point of inflexion $x = 0$, $z = 0$; while no part of the curve lies between that ordinary tangent and the tangent to the Hessian $x = 0$, but a convex part lies between the last-named tangent and the third ordinary tangent to u,

$$\sqrt{d}\,x + \sqrt{3c}\,z = 0.$$

The accompanying figure is the locus of

$$u \equiv -16x^3 + 6y^2 + 3z^2x,$$

viz., in the above formula, $-a = d = 16$, $b = 2$, $c = 1$, the sides of the triangle (ABC) of reference being as $4 : 3 : 2$. The longest and shortest sides are the segments on the inflexion tangents to u and its Hessian (which, to a numerical factor, is)

$$w \div -2 = 16x^2z - 32y^2x + z^3 = 0,$$

respectively, between a point of inflexion and its harmonic polar, the

mean side CA being the segment on the harmonic polar of B (zx) between those tangents.

Curves have been but little traced through their trilinear, or other homogeneous equations. In the present case, two of the three corners of the triangle of reference, viz., $x=0,\ z=0;\ x=0,\ y=0$, being on the curve, it is easy to determine other points as the intersections of lines through those two corners, drawing such lines through their ascertained divisions of the sides of the triangle of reference. If the line

$$z = kx$$

divides that side which is a segment of $y=0$ into parts z_0, x_0, then, plainly, the coordinates of the point of division will be

$$z : x = z_0 : 2x_0,$$

whence

$$z_0 = 2kx_0,$$

$$z_0 = \frac{2k}{2k+1}\,CA.$$

Then (iv.) the lines drawn to $y=0,\ x=0$ from the points in which (v.) meets the cubic u are

$$\sqrt{6k}\,y = \pm\sqrt{16-3k^2}\,x,$$

and, if these divide the side (AB) which is a segment of $z=0$ into the two parts y_1, x_1,

$$y : x = 2y_1 : 3x_1,$$

whence

$$x_1 : y_1 = 2\sqrt{6k} : \pm 3\sqrt{16-3k^2},$$

$$x_1 = 2\sqrt{6k}\,AB/(2\sqrt{6k}\pm 3\sqrt{16-3k^2}).$$

One of the values of x_1 having been calculated, the other may be constructed with the ruler by the property of the harmonic division.

It will be remarked in the figure, that the three harmonic polars CA, $C'A'$, $C''A''$ of the inflexion points B, B', B'' have a common intersection O. This is a property not confined to the class of cubics now under consideration, but common to all cubics: viz., the property that "the harmonic polars of three collinear points of inflexion on any cubic meet in one point," but it is beside the object of the present paper to discuss this question generally. In the particular case of the figure, the reason is at once evident from the consideration of the harmonic division of lines.

To determine the asymptotes of (u),

$$-16x^3+6y^2z+3z^2x=0,$$

substitution may be made in the general formula (12), or, as involving less arithmetical calculation, investigate the question from elementary considerations, thus:

The line at infinity being, in the present case,

$$2x+3y+4z=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

eliminating y between it and u,

$$-48x^3+8x^2z+41z^2x+32z^3=0,$$

the real root of which is, very nearly,

$$x/z = 101/80,$$

whence (13) $\qquad y/z = -87/40.$

For these values of the coordinates the first differential coefficients of u are

$$u_1 : u_2 : u_3 = -21202 : -6860 : 9589.$$

The real asymptote, therefore, coincides nearly with

$$21202x+6860y-9589z=0.$$

As many points, other than the contacts particularly considered above, as desired may now be obtained by giving k different numerical values in the equations, p. 387.

The inflexional tangents to w $(T' \ldots)$ touch u on the lines $BC'C''$ (viii.) ...

To trace the Hessian

$$w/2 \equiv z^3+16x^2z-32y^2x=0 \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

making, as above, $\qquad z=kx,$

$$32y^2/x^2 = k(k^2+16);$$

so that k can have no negative value—viz., the curve lies wholly within one angle contained by tangents to the primitive curve and its Hessian at a point of inflexion—and consists of a single serpentine branch, with a single real asymptote.

Proceeding, as for u, to determine this asymptote,

$$9z^3-512z^2x-368x^2z-128x^3=0,$$

the real root of which is, very nearly,

$$z-57{\cdot}605x=0,$$

from which $$y = -76{\cdot}8x.$$

Substituting these values of the coordinates in the first differential coefficients of w, viz.,

$$32\,(zx-y^2), \quad -64xy, \quad 16x^2+3z^2,$$

they are found to be proportional to the coefficients in

$$-1521x+40y+648z = 0,$$

the locus of which is therefore the real asymptote.

The three real inflexional tangents to the Hessian, w, are $x = 0$, BC, touching it at the point $x=0$, $z=0$, and the two lines, $B'C'$, $B''C''$, the equations to which are given in general terms above (v., p. 385).

The inflexional tangents to u $(T\ldots)$ touch w on the lines (xii.) $BA'A''\ldots$.

Making in w, (14), $$z = \lambda y,$$

$$16\lambda x^2-32xy+\lambda^3y^2 = 0,$$

which will have equal roots, if

$$\lambda^4 = 16,$$

$$\lambda = \pm 2;$$

viz., $$z = \pm 2y,$$

are the real tangents to w from the point $y = 0$, $z = 0$, besides the line $z = 0$ itself. Between the values $\lambda = 2$, $\lambda = -2$, no line through the point $y = 0$, $z = 0$ meets w in any real point; while, for all values of λ greater than 2 and less than -2, every line through $y = 0$, $z = 0$ meets the Hessian w in two distinct real points. By giving λ different numerical values, as many more real points on w as desired may be constructed.

The equation to P in line coordinates being (3) for the values

$$a = -16, \quad b = 2, \quad c = 1,$$

$$P = -\alpha^3+16\beta^2\gamma+16\gamma^2\alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

$\gamma = 0$ will be a cusp, and $\alpha = 0$, $\gamma = 0$ the corresponding cuspidal tangent.

Analogously, the other points of contact of the inflexional tangents to the Hessian with u will be cusps on P, and the connectors of those points with the corresponding points of contact with the Hessian of the inflexional tangents to u will be the cuspidal tangents to P. Thus the real tricuspidal part of P is inscribed in the convex part of u.

Making $\beta = 0$ in (15),

$$\alpha = 0 \quad \text{or} \quad \alpha \pm 4\gamma = 0;$$

viz., $z = 0$, *i.e.*, the inflexional tangent of u is a tangent to P at its contact with the Hessian; or, what is the same thing, P touches the Hessian at the points at which it is touched by the inflexional tangents to u. P, therefore, does not meet the Hessian at any real point other than the three points of contact of the real inflexional tangents to u.

Since (15)

$$\partial P/\partial\alpha = -3\alpha^2+16\gamma^2, \quad \partial P/\partial\beta = 32\beta\gamma, \quad \partial P/\partial\gamma = 16\,(\beta^2+2\gamma\alpha),$$

eliminating β between (15) and

$$2\,(-3\alpha^2+16\gamma^2)+96\beta\gamma+64\,(\beta^2+2\gamma\alpha) = 0 \quad \ldots\ldots\ldots(16),$$

$$(2\alpha^3-3\alpha^2\gamma+32\gamma^2\alpha+16\gamma^3)^2+144\alpha\gamma^3\,(-\alpha^2+16\gamma^2) = 0$$

determines the ratios of the coefficients of x and z in the equations to the asymptotes of P.

The real roots of this equation are, very nearly,

$$\alpha : \gamma = 8 : -5,$$

and $$= 1 : -13.$$

To determine β, eliminate β^2 between (15), (16), giving

$$\beta = (-2\alpha^3+3\alpha^2\gamma-32\alpha\gamma^2-16\gamma^3)\,/\,48\gamma^2,$$

which for the former of the two values of $\alpha : \gamma$ gives

$$\beta : \gamma = 4 : 3,$$

and for the latter $$\beta : \alpha = 3 : 1.$$

Thus the real asymptotes nearly coincide with

$$24x-20y-15z = 0,$$

$$x+3y-13z = 0.$$

What has been shown is sufficient to enable one to draw P with moderate correctness; for more detail it is easier to have recourse to its equation in point coordinates (7, 8), which for the values $a = -16$, $b = 2$, $c = 1$ is

$$Y^3+(1-18k^2-27k^4)\,Y+4k\,(1+27k^4) = 0 \ldots\ldots\ldots\ldots(17),$$

if
$$Y = y^2/2x^2 + z/x,$$
$$k = z/4x.$$

The discriminant of (17), considered as a cubic in Y, is
$$(1+18k^2-27k^4)^3;$$
the real values of k for which this vanishes being
$$k = \pm 2p/r^3,$$
$$= \pm r/3q$$
(where, as before, $p, q = \sqrt{3} \pm 1$, $r = \sqrt{2\sqrt{3}}$),

or
$$4rx = \pm 3qz \ldots\ldots\ldots\ldots (18).$$

The equal roots of (17) for this value of k are
$$Y = \mp 2p/r,$$
whence
$$y^2/2x^2 = 8p/r^3 + 2p/r$$
$$= 2p\,(4+2\sqrt{3})/r^3$$
$$= 2p^3/r^3,$$
$$r^3y^2 = 4p^3x^2 \ldots\ldots\ldots\ldots (19).$$

Equations (18), (19) determine the two contacts with u of the real inflexion tangents to w other than $x = 0$ (v.), (viii.), p. 385.

At the points in which $y = 0$ meets P, as appears from (8),
$$x = 0, \quad z = 0, \quad 4x+z = 0, \quad 4x-z = 0,$$
all meet P in two coincident points, viz., a cusp in the case of the first of these four lines, and an ordinary point of contact in the case of the other three. Recollecting that $y = 0$ is the harmonic polar of $x = 0$, $z = 0$, it is thus shown, generally, that the connectors of the points (not being cusps) in which the harmonic polars of the points of inflexion meet P with the latter points, touch P at the former points.

The Quippian of u being the Pippian of w, it does not seem necessary to go into much detail in explaining its figure, further than to point out the differences. Though having two real asymptotes, Q wants the hyperbolic branches of P; an arm of one of the pair of cusps (A', A'') which touches the ovoid part of u forming, with an arm of the third cusp (A), the parts having a common asymptote.

For the example, $u = -16x^3 + 6y^2z + 3z^2x$,

The discriminant of (

which vanishes if

where, as before,

—

On the Small Wave-Mot

Gravity.

[*Read*

In determining the possibl
motionless homogeneous inco
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at once, this is necessarily so in

[illegible]

the equations to Q are

$$Q/256 = 16\gamma^3 + 3a^2\gamma - 3a\beta^2 = 0,$$

and, in point coordinates,

$$Y^3 + 3(3k^4 - 6k^2 + 1)Y + 12k(3k^4 + 1) = 0 \quad \ldots\ldots\ldots\ldots (20);$$

if

$$Y = 8y^2/z^2 - 16x/z,$$

$$k = 4x/z.$$

The discriminant of (20) is, to a numerical factor,

$$(3k^4 + 6k^2 - 1)^3,$$

which vanishes if $\quad k = \pm q/r,$

where, as before, $\quad q = \sqrt{3} - 1,$

$$r = \sqrt{2\sqrt{3}}.$$

On the Small Wave-Motions of a Heterogeneous Fluid under Gravity. By W. Burnside.

[*Read June 13th*, 1889.]

In determining the possible small wave-motions of an otherwise motionless homogeneous incompressible fluid under gravity, it is always assumed that the motion is irrotational, and, as may be seen at once, this is necessarily so in the approximate problem.

For the equations of motion are

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t},$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial w}{\partial t} - g,$$

and if $\quad u = u_0 \sin(mx - nt),$

$$w = w_0 \cos(mx - nt),$$

... are unequal ... and ...
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where u_0, w_0 are functions of z, it may be verified at once that

$$\frac{\partial}{\partial z} - \frac{\partial w}{\partial x} = 0.$$

If, however, the fluid, still incompressible, be of varying density, it will no longer be generally possible to satisfy the above equations by assuming the motion irrotational; and indeed, since for a given wave-length such a fluid would still have an infinite number of degrees of freedom, it is otherwise obvious that an irrotational wave-motion, if it exists, would be only one out of an infinite number of possible wave-motions.

There is little difficulty in determining directly from the above equations the possible small wave-motions of such a fluid; but the following process in which the continuously varying density is regarded as the limit of a number of finite strata of finitely differing densities, leads to the result in what seems a rather more elegant manner.

At the surface of separation of two different fluids in which a small wave-motion is taking place, the geometrical as well as the dynamical conditions are only satisfied approximately, and moreover there is finite slip of the fluids one over the other. It might be expected that, in passing to the limit, there would result a motion in which the equation of continuity would be only satisfied approximately, or in which at certain depths there would be finite tangential slip; but, as will be seen, this is not the case.

Take the successive strata each of depth h, and suppose that ρ_r is the density of the r^{th} stratum from the top; the velocity-potential in this stratum will be of the form*

$$\phi_r = (A_r \cosh mz + B_r \sinh mz) \cos m (x - Vt),$$

and the conditions to be satisfied are

$$\left.\begin{aligned} \frac{\partial \phi_{r+1}}{\partial z} &= \frac{\partial \phi_r}{\partial z}, \\ \rho_{r+1}\left(\frac{\partial^2 \phi_{r+1}}{\partial t^2} - g\frac{\partial \phi_{r+1}}{\partial z}\right) &= \rho_r\left(\frac{\partial^2 \phi_r}{\partial t^2} - g\frac{\partial \phi_r}{\partial z}\right), \\ \text{when} \qquad z &= rh; \end{aligned}\right\} \begin{matrix} \text{from } r = 1 \\ \text{to } r = n-1 \end{matrix}$$

$$\frac{\partial^2 \phi_1}{\partial t^2} - g\frac{\partial \phi_1}{\partial z} = 0, \qquad \text{when } z = 0;$$

$$\frac{\partial \phi_n}{\partial z} = 0, \qquad \text{when } z = nh.$$

Using difference notation for r, these are the same as

$$\left.\begin{aligned}&\sinh mrh\,\Delta A_r+\cosh mrh\,\Delta B_r=0,\\&\cosh mrh\,\Delta(A_r\rho_r)+\sinh mrh\,\Delta(B_r\rho_r)\\&\quad+\frac{g}{mV^2}(\sinh mrh\,A_r+\cosh mrh\,B_r)\,\Delta\rho_r=0,\end{aligned}\right\}\begin{array}{l}\text{from } r=1\\ \text{to } r=n-1\end{array}$$

$$A_1+\frac{g}{mV^2}B_1=0,$$

$$\sinh mnh\,A_n+\cosh mnh\,B_n=0.$$

Suppose that h now becomes infinitesimal, and n infinite, and put

$$rh=\zeta,\quad h=d\zeta,\quad nh=H,\quad \text{the total depth.}$$

The equations become

$$\sinh m\zeta\frac{dA}{d\zeta}+\cosh m\zeta\frac{dB}{d\zeta}=0,$$

$$\cosh m\zeta\frac{d(A\rho)}{d\zeta}+\sinh m\zeta\frac{d(B\rho)}{d\zeta}$$
$$+\frac{g}{mV^2}(\sinh m\zeta\,A+\cosh m\zeta\,B)\frac{d\rho}{d\zeta}=0,$$

with the conditions

$$A+\frac{g}{mV^2}B=0,\qquad\text{when }\zeta=0,$$

$$\sinh m\zeta\,A+\cosh m\zeta\,B=0,\quad\text{when }\zeta=H.$$

By writing

$$\sinh m\zeta\,A+\cosh m\zeta\,B=\alpha,$$
$$\cosh m\zeta\,A+\sinh m\zeta\,B=\beta,$$

these equations take the simpler form

$$\frac{d\alpha}{d\zeta}-m\beta=0,$$

$$\frac{d\beta}{d\zeta}-m\alpha+\frac{1}{\rho}\frac{d\rho}{d\zeta}\left(\beta+\frac{g}{mV^2}\alpha\right)=0,$$

with the conditions

$$\beta+\frac{g}{mV^2}\alpha=0,\quad\text{when }\zeta=0;$$

$$\alpha=0,\quad\text{when }\zeta=H.$$

By expressing that the two conditions can be simultaneously satisfied by the solutions of the differential equations, a transcendental equation for V the wave-velocity will be obtained.

Before going on to consider a special law of variation of the density at length, it may at once be noticed that the irrotational motion, if it exists, must be independent of $\frac{d\rho}{d\zeta}$. For, when the motion is irrotational, A and B must be constant; and hence the equations for α and β reduce to

$$\beta + \frac{g}{mV^2}\alpha = 0 \quad \text{always,}$$

and

$$\alpha = 0 \quad \text{when } \zeta = H.$$

Hence

$$B + \frac{g}{mV^2}A = 0,$$

$$A + \frac{g}{mV^2}B = 0,$$

and

$$A\sinh mH + B\cosh mH = 0,$$

which are only consistent when H is infinite, and then give

$$A + B = 0,$$

$$V^2 = \frac{g}{m} = \frac{g\lambda}{2\pi},$$

where λ is the wave-length.

By considering the nature of the motion at the bottom, it is tolerably obvious that, when the depth is finite, there can be no irrotational motion.

The simplest special case that can be considered is when $\rho \propto \rho_0 e^{m'z}$, so that

$$\frac{1}{\rho}\frac{d\rho}{d\zeta} = m', \text{ a constant.}$$

In this case the solutions of the equations for α and β will be of the forms

$$\alpha = \alpha_0 e^{p\zeta},$$

$$\beta = \beta_0 e^{p\zeta};$$

and, on substituting these values in the differential equations, there

results

$$p\alpha_0 - m\beta_0 = 0,$$

$$\left(\frac{gm'}{mV^2} - m\right)\alpha_0 + (m'+p)\,\beta_0 = 0,$$

and hence for p the equation

$$p\,(p+m') + \frac{gm'}{V^2} - m^2 = 0.$$

If p_1, p_2 are the roots of this quadratic, the complete solutions of the differential equations are

$$\alpha = a_1 e^{p_1\zeta} + a_2 e^{p_2\zeta},$$

$$m\beta = p_1 a_1 e^{p_1\zeta} + p_2 a_2 e^{p_2\zeta},$$

and the equations of condition become

$$a_1 e^{p_1 H} + a_2 e^{p_2 H} = 0,$$

and

$$\left(p_1 + \frac{g}{V^2}\right) a_1 + \left(p_2 + \frac{g}{V^2}\right) a_2 = 0\,;$$

whence

$$e^{(p_1 - p_2)H} = \frac{p_1 + \frac{g}{V^2}}{p_2 + \frac{g}{V^2}},$$

or

$$\frac{2g}{V^2} - m' = (p_1 - p_2)\coth\frac{p_1 - p_2}{2}H,$$

where the right-hand side is a symmetric function of the roots of the equation for p.

The velocities at any point of the fluid are given by

$$u = -m\,(A\cosh mz + B\sinh mz)\sin m\,(x - Vt),$$

$$w = m\,(A\sinh mz + B\cosh mz)\cos m\,(x - Vt),$$

or

$$u = -m\beta\sin m\,(x - Vt),$$

$$w = m\alpha\cos m\,(x - Vt),$$

where now, in the values just found for α and β, z is written for ζ.

If the depth H becomes infinite, these forms for u and w show that p_1 and p_2 must be both negative. When this condition is satisfied, the motion diminishes indefinitely with the depth independently of

the ratio a_1/a_2, and by suitably assigning the value of this ratio the surface condition can be satisfied whatever p_1 and p_2 may be.

Hence, in this case, p_1 and p_2 may be *any* two negative quantities satisfying the relation

$$p_1+p_2=-m',$$

and the velocity of propagation is given by

$$V^2=\frac{gm'}{p_1p_2+m^2}.$$

For a given value of m, the greatest of these values of V^2 is gm'/m^2, so that the irrotational mode would seem to stand by itself, and not to occur as one limiting case of the rotational modes.

It might have been expected, *à priori*, that in the case of infinite depth, the infinity of possible motions would have been of a higher order than in the case of finite depth; and this is seen to be so, the former case being comparable with the infinity of possible positions of points on a line, and the latter with the infinite series of roots of a transcendental equation.

On some Rings of Circles connected with a Triangle, and the Circles which cut them at Equal Angles. *By* W. W. Taylor.

[*Read June* 13*th*, 1889.]

If any three circles be placed in contact, the lines joining their points of contact A, B, C form a triangle. Hence it would appear that such three circles must play an important part in the geometry of the triangle. They may be defined, with reference to the triangle ABC, as the circles* that touch two of the radii of the circle ABC at the angular points of the triangle. We will proceed to find their equations, and discuss their properties, and those of certain associated circles and triangles.

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* These circles have been called the ex-cosine circles of the triangle ABC. (W. E. Johnson's "Trigonometry," § 194.)

[The centres of the same set of circles form a second triangle, an the circles may be defined, with reference to that triangle, as the circl which are orthogonal to the inscribed circle, and have their centre at the angular points of the triangle. The properties obtained fo the first set may be transformed so as to suit the second set of circle by means of three formulæ like

$$\frac{\delta}{d} = \frac{R}{rs}\left(-\alpha\cos^2\frac{A}{2}+\beta\cos^2\frac{B}{2}+\gamma\cos^2\frac{C}{2}\right),$$

where α, β, γ are the trilinear coordinates of a point referred to th first triangle ABC, and δ, ϵ, ξ new coordinates referred to the side d, e, f, *i.e.*, EF, FD, DE, where D, E, F are the feet of the perpen diculars from A, B, C on the opposite sides.]

If the equation

$$\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C$$
$$-(l\alpha+m\beta+n\gamma)(\alpha\sin A+\beta\sin B+\gamma\sin C) = 0 \quad\text{(1)}$$

represent a circle orthogonal to the circle ABC, the straight lin whose equation is $l\alpha+m\beta+n\gamma = 0$ must be the polar of O the centr of the circle ABC, *i.e.*, of the point $R\cos A$, $R\cos B$, $R\cos C$. Th condition for this is

$$1-l\cos A-m\cos B-n\cos C = 0 \quad\text{(2)}$$

and the equation of the circle becomes

$$(l\cos A+m\cos B+n\cos C)\,(\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C)$$
$$-(l\alpha+m\beta+n\gamma)(\alpha\sin A+\beta\sin B+\gamma\sin C) = 0,$$

which can be written in the form

$$l\sin A\,(\alpha^2-\beta\gamma\cos A+\gamma\alpha\cos B+\alpha\beta\cos C)$$
$$+m\sin B\,(\beta^2+\beta\gamma\cos A-\gamma\alpha\cos B+\alpha\beta\cos C)$$
$$+\;n\sin C\,(\gamma^2+\beta\gamma\cos A+\gamma\alpha\cos B-\alpha\beta\cos C) = 0.$$

If the common chord be the line $\alpha = 0$, then $l = +\sec A$, and the equation of the circle orthogonal to the circle ABC is

$$\alpha^2-\beta\gamma\cos A+\gamma\alpha\cos B+\alpha\beta\cos C = 0.$$

We will denote the left-hand side of this by the letter A_1; the equations of the three circles that touch OB, OC; OC, OA; OA, OB at the angular points will be

$$\left.\begin{aligned} A_1 &\equiv \alpha^2-\beta\gamma\cos A+\gamma\alpha\cos B+\alpha\beta\cos C = 0\\ B_1 &\equiv \beta^2+\beta\gamma\cos A-\gamma\alpha\cos B+\alpha\beta\cos C = 0\\ C_1 &\equiv \gamma^2+\beta\gamma\cos A+\gamma\alpha\cos B-\alpha\beta\cos C = 0\end{aligned}\right\} \quad\text{(3)}$$

The equation of any other circle orthogonal to the circle ABC will be

$$l \sin A \,.\, A_1 + m \sin B \,.\, B_1 + n \sin C \,.\, C_1 = 0 \quad\ldots\ldots\ldots\ldots\ldots(4),$$

where $l\alpha + m\beta + n\gamma = 0$ is the equation of their common chord.

It will be observed that, as each of the three circles A_1, B_1, C_1 touches the other two, and as there must be a pair of circles that each touches, these three circles, A_1. B_1, C_1, must form a ring in the sense of Mr. H. M. Taylor's paper on "The Porism of the Ring of Circles touching Two Circles," *Messenger of Mathematics*, Vol. VII., 1878.)*

We shall accordingly refer to them as the 3-ring circles A_1, B_1, C_1.

The centre of the circle A_1 is most easily found by taking the tangents to the circumscribed circle at B, C, and finding their intersection.

It is obviously $$\frac{\alpha}{-a} = \frac{\beta}{b} = \frac{\gamma}{c} = \frac{2\Delta}{-a^2+b^2+c^2}.$$

Hence we see that the centres of these circles are the "associates of the Lemoine point."

To find the length of the intercept cut off on the side AC of the triangle by the circle A_1, we have the equations

$$\beta = 0, \quad \alpha = -\gamma \cos B \text{ by (3), and } a\alpha + c\gamma = 2\Delta,$$

whence $$\alpha = -a \sin C \cos B \sec A,$$

and the intercept

$$CP = \pm\, \alpha \operatorname{cosec} C = \pm\, a \cos B \sec A\,;$$

similarly the intercept

$$BQ = \pm\, a \cos C \sec A,$$

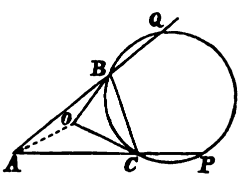
and $$BC : CP : BQ = \cos A : \pm \cos B : \pm \cos C.$$

It is worthy of notice that the six points corresponding to P, Q lie on the conic

$$\Sigma\alpha^2 + \Sigma\beta\gamma\,(\cos A + \sec A) = 0,$$

and also that

$$A_1B_1 + B_1C_1 + C_1A_1 \equiv \{\Sigma\,(\beta\gamma \sin A)\}^2.$$



* Compare also a paper in the same volume "On the Ring of Circles touching Two Circles, and kindred Porisms."

LEMMA I.

If α, β, γ be the trilinear coordinates of a point, the equation of circle being expressed in the form

$$\phi(\alpha, \beta, \gamma) \equiv \Sigma\beta\gamma \sin A - (l\alpha + m\beta + n\gamma)\,\Sigma\alpha \sin A = 0,$$

or in the form

$$(l, m, n) \equiv \Sigma a\beta\gamma - (l\alpha + m\beta + n\gamma)\,\Sigma a\alpha = 0,$$

the coordinates of its centre are $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$, where

$$\bar{\alpha} = R(-l + m\cos C + n\cos B + \cos A),$$

$$\bar{\beta} = R(-m + n\cos A + l\cos C + \cos B),$$

$$\bar{\gamma} = R(-n + l\cos B + m\cos A + \cos C),$$

and its radius is ρ, where

$$\rho^2 = R^2(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C$$
$$-2l\cos A - 2m\cos B - 2n\cos C + 1).$$

The condition that the centre of the circle (l, m, n) is the pol of the line at infinity gives the following equations to find it coordinates :—

$$\left\{c\beta + b\gamma - a(l\alpha + m\beta + n\gamma) - l(a\alpha + b\beta + c\gamma)\right\}/a$$
$$= \left\{a\gamma + c\alpha - b(l\alpha + m\beta + n\gamma) - m(a\alpha + b\beta + c\gamma)\right\}/b$$
$$= \left\{b\alpha + a\beta - c(l\alpha + m\beta + n\gamma) - n(a\alpha + b\beta + c\gamma)\right\}/c.$$

Let each of these $= X - (l\alpha + m\beta + n\gamma)$. Then we have the equation

$$c\beta + b\gamma - l(a\alpha + b\beta + c\gamma) = aX,$$

$$a\gamma + c\alpha - m(a\alpha + b\beta + c\gamma) = bX,$$

$$b\alpha + a\beta - n(a\alpha + b\beta + c\gamma) = cX.$$

Then, eliminating β, γ, we obtain the equation

$$\alpha \begin{vmatrix} -al & c-bl & b-cl \\ c-am & -bm & a-cm \\ b-an & a-bn & -cn \end{vmatrix} = X \begin{vmatrix} a & c-bl & b-cl \\ b & -bm & a-cm \\ c & a-bn & -cn \end{vmatrix}.$$

The coefficient of X becomes, on expansion,

$$2abc\,(\cos A - l + m\cos C + n\cos B).$$

Therefore we have

$$\alpha : \beta : \gamma : a\alpha + b\beta + c\gamma$$

$$= \cos A - l + m\cos C + n\cos B : \cos B - m + n\cos A + l\cos C$$

$$: \cos C - n + l\cos B + m\cos A : a\cos A + b\cos B + c\cos C;$$

but $$a\alpha + b\beta + c\gamma = aR\cos A + bR\cos B + cR\cos C,$$

where R is the radius of the circle ABC;

therefore
$$\left.\begin{aligned} \alpha &= R\,(\cos A - l + m\cos C + n\cos B) \\ \beta &= R\,(\cos B - m + n\cos A + l\cos C) \\ \gamma &= R\,(\cos C - n + l\cos B + m\cos A) \end{aligned}\right\} \quad \ldots\ldots\ \ldots\ldots\ldots (5);$$

and, if ρ is the radius of the circle $\phi = 0$,

$$\rho^2\phi\,(-1, \cos C, \cos B) = -\phi\,(\alpha_0, \beta_0, \gamma_0);$$

whence, by substituting the above values, we obtain

$$\rho^2 = R^2\,(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C$$
$$-2l\cos A - 2m\cos B - 2n\cos C + 1) \ \ldots\ldots (6).$$

LEMMA II.

We will now proceed to find the cosine of the angle θ, at which two circles (l_1, m_1, n_1), (l_2, m_2, n_2) whose equations are expressed in the form (1) cut one another.

If the centres of these circles be O_1, O_2, and their radii R_1, R_2,

$$\cos\theta = \frac{O_1O_2^2 - R_1^2 - R_2^2}{2R_1R_2},$$

and $$R_1 = \frac{8R^3}{abc}\,\phi_1,$$

where ϕ_1 denotes the value of the expression

$$\Sigma\beta\gamma\sin A - (\Sigma l\alpha)(\Sigma\alpha\sin A),$$

when for α, β, γ we substitute the coordinates of the centre.

Let the equations of the circles whose radii are R_1, R_2, and centres O_1, O_2, be

$$\phi_1\,(\alpha, \beta, \gamma) \equiv \Sigma\beta\gamma\sin A - (\Sigma l_1\alpha)(\Sigma\alpha\sin A),$$

$$\phi_2\,(\alpha, \beta, \gamma) \equiv \Sigma\beta\gamma\sin A - (\Sigma l_2\alpha)(\Sigma\alpha\sin A).$$

Then the equation of a circle whose centre is O_1 and radius O_1O_2 w be

$$\phi_1(\alpha, \beta, \gamma) + h(\Sigma\alpha \sin A)^2 = 0;$$

and, as $O_2(\alpha_2, \beta_2, \gamma_2)$ is on this circle,

$$\phi_1(\alpha_2, \beta_2, \gamma_2) + h(\Sigma\alpha \sin A)^2 = 0;$$

and
$$O_1O_2^2 = \frac{8R^3}{abc}\{\phi_1(\alpha_1, \beta_1, \gamma_1) + h(\Sigma\alpha \sin A)^2\}$$

$$= \frac{8R^3}{abc}\{\phi_1(\alpha_1, \beta_1, \gamma_1) - \phi_1(\alpha_2, \beta_2, \gamma_2)\},$$

and

$$\cos\theta = \frac{\phi_1(\alpha_1, \beta_1, \gamma_1) - \phi_1(\alpha_2, \beta_2, \gamma_2) - \phi_1(\alpha_1, \beta_1, \gamma_1) - \phi_2(\alpha_2, \beta_2, \gamma_2)}{\sqrt{\{\phi_1(\alpha_1, \beta_1, \gamma_1)\,\phi_2(\alpha_2, \beta_2, \gamma_2)\}}}$$

$$= \frac{-2\Sigma\beta_2\gamma_2 \sin A + \{\Sigma(l_1+l_2)\,\alpha_2\}\,\Sigma\alpha_2 \sin A}{\sqrt{\{\phi_1(\alpha_1, \beta_1, \gamma_1\}\,\{\phi_2(\alpha_2, \beta_2, \gamma_2)\}}};$$

whence, by substituting the values for $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, we obtain

$$\cos\theta = \frac{-1 - \Sigma l_1 l_2 + \Sigma \cos A\,(l_1 + l_2 + m_1 n_2 + m_2 n_1)}{\sqrt{\{[1+\Sigma l_1^2 - 2\Sigma \cos A\,(l_1 + m_1 n_1)]\,[1+\Sigma l_2^2 - 2\Sigma \cos A\,(l_2 + m_2 n_2)]}} \quad \text{.........(7}$$

Hence, if the circle (l, m, n) make the same angle θ with each the three circles

$$(\sec A, 0, 0), \quad (0, \sec B, 0), \quad (0, 0, \sec C),$$

we must have, by (7),

$$\cos\theta\,\sqrt{[1+\Sigma l^2 - 2\Sigma \cos A\,(l+mn)]}$$

$$= \{-1 - l\sec A + \cos A\,(l + \sec A) + \cos B\,(m + n\sec A)$$
$$+ \cos C\,(n + m\sec A)\}/\sqrt{[1+\sec^2 A - 2]}$$

$$= \sec A\{-l(1-\cos^2 A) + m(\cos C + \cos A\cos B)$$
$$+ n(\cos B + \cos A \cos C)\}/\tan$$

$$= \sec A\{-l\sin^2 A + m\sin A\sin B + n\sin A\sin C\}/\tan A$$

$= -l\sin A + m\sin B + n\sin C$, and in like manner

$$= +l\sin A - m\sin B + n\sin C$$

$$= +l\sin A + m\sin B - n\sin C;$$

therefore
$$l\sin A = m\sin B = n\sin C.$$

Let each of these $=p$. Then, substituting for l, m, n their values in terms of p,

$$\cos\theta\,\sqrt{\left[1+p^2\Sigma\operatorname{cosec}^2 A-2p\Sigma\cot A-2p^2\Sigma\cos A\operatorname{cosec} B\operatorname{cosec} C\right]}=p;$$

or, putting $\qquad \cot A+\cot B+\cot C=\cot\omega,$

$$\cos\theta\sqrt{\left[1+p^2\operatorname{cosec}^2\omega-2p\cot\omega-4p^2\right]}=p,$$

$$\cos\theta=\frac{p}{\sqrt{\{(1-p\cot\omega)^2-3p^2\}}}=\frac{1}{\sqrt{(\lambda^2-3)}},$$

where $\qquad \lambda=\dfrac{1}{p}-\cot\omega.$

Making these substitutions, the equation of the circle (l, m, n) reduces to the form

$$\Sigma\beta\gamma\sin A-p\,(\Sigma\alpha\operatorname{cosec} A)\,(\Sigma\alpha\sin A)=0,$$

or $\qquad -p\Sigma\alpha^2+\Sigma\beta\gamma\sin A-p\Sigma\beta\gamma\operatorname{cosec} B\operatorname{cosec} C\,(\sin^2 B+\sin^2 C)=0,$

or $\qquad p\Sigma\alpha^2-\Sigma\beta\gamma\sin A\,(1-p\cot\omega)+p\Sigma\beta\gamma\cos A=0,$

$$\Big[\text{since}\quad \tfrac{1}{2}\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C\,(\sin^2 A+\sin^2 B+\sin^2 C)$$
$$+\tfrac{1}{2}\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C\,(\sin^2 B+\sin^2 C-\sin^2 A)$$
$$=\cot\omega+\operatorname{cosec} A\cos A\Big],$$

and substituting λ for $(1-p\cot\omega)/p$, we obtain for the equation of our circle

$$\Sigma\alpha^2+\Sigma\beta\gamma\cos A-\lambda\Sigma\beta\gamma\sin A=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

and, if this touch the circles A_1, B_1, C_1,

$$\sqrt{(\lambda^2-3)}=\sec\theta=1,$$

$$\lambda=\pm 2.$$

The formulæ for the centre of a circle become in this case

$$\left.\begin{aligned}\alpha&=R\,(\sin A+\lambda\cos A)\,/\,(\lambda+\cot\omega)\\ \beta&=R\,(\sin B+\lambda\cos B)\,/\,(\lambda+\cot\omega)\\ \gamma&=R\,(\sin C+\lambda\cos C)\,/\,(\lambda+\cot\omega)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

With the same substitutions

$$\rho^2=R^2\,(\Sigma l^2-2\Sigma mn\cos A-2\Sigma l\cos A+1)$$
$$=R^2\,(\Sigma p^2\operatorname{cosec}^2 A-2\Sigma p^2\cos A\operatorname{cosec} B\operatorname{cosec} C-2\Sigma p\cot A+1)$$
$$=R^2\,(p^2\operatorname{cosec}^2\omega-2p^2\Sigma\,(1-\cot A\cot B)-2p\cot\omega+1)$$

$$= R^2 (p^2 \operatorname{cosec}^2 \omega - 4p^2 - 2p \cot \omega + 1)$$
$$= R^2 (p \cot \omega - 1)^2 - 3p^2$$
$$= R^2 (\lambda^2 - 3) p^2$$
$$= R^2 (\lambda^2 - 3) / (\lambda + \cot \omega)^2.$$
$$\rho = \pm R \sqrt{(\lambda^2 - 3)} / (\lambda + \cot \omega) \quad \text{.......} \quad (10).$$

The cosine of the angle between this circle (8) and a ring circle (l, m, n) for which $l \cos A + m \cos B + n \cos C = 1$, is

$$\frac{-1 - \Sigma lp \operatorname{cosec} A + \Sigma \{ l \cos A + p \cot A + pl (\cos B \operatorname{cosec} C + \cos C \operatorname{cosec} B) \}}{\sqrt{\{ [(l^2 + m^2 + n^2) - (l \cos A + m \cos B + n \cos C)^2 - 2\Sigma mn \cos A] (\lambda^2 - 3) p^2 \}}}$$
$$= \frac{-1 + \Sigma lp (\sin A - \cos A \cot \omega) + 1 + p \cot \omega}{p \sqrt{\{ (\Sigma l^2 \sin^2 A - 2\Sigma mn \sin B \sin C)(\lambda^2 - 3) \}}}$$
$$= \frac{\Sigma l \sin A}{\sqrt{\{ (\Sigma l^2 \sin^2 A - 2\Sigma mn \sin B \sin C)(\lambda^2 - 3) \}}} \quad \text{.......} \quad (11).$$

The series of circles included in the equation

$$\Sigma \alpha^2 + \Sigma \beta\gamma \cos A - \lambda \Sigma \beta\gamma \sin A = 0$$

has been discussed by Professor P. H. Schoute, in Vol. III., Series 3, of the *Verslagen en Mededeelingen* of the *Koninklijke Akademie van Wetenschappen*, Amsterdam, from an entirely different point of view. He shows that, when a point P moves so that, if D, E, F be the feet of the perpendiculars from it on the sides of the triangle, the Brocard angle of the triangle DEF is constant, the locus of P is a circle of the above series, and λ is the cotangent of the said Brocard angle.

He has shown that this series of circles includes, as particular cases: the Brocard circle ($\lambda = \cot \omega$); the imaginary circle whose equation is

$$\Sigma \alpha^2 + \Sigma \beta\gamma \cos A = 0, \quad (\lambda = 0);$$

the circle ABC ($\lambda = \infty$); the Lemoine line ($\lambda = -\cot \omega$), and the isodynamic points ($\lambda = \pm \sqrt{3}$).

Of these six particular results the first three are obvious by a comparison of the equation No. 8 with the equations of the other circles, and the last three can be obtained by making the centre of the circle lie on the locus.

The condition for this is

$$\begin{vmatrix} 2 & \cos C - \lambda \sin C & \cos B - \lambda \sin B \\ \cos C - \lambda \sin C & 2 & \cos A - \lambda \sin A \\ \cos B - \lambda \sin B & \cos A - \lambda \sin A & \end{vmatrix} = 0,$$

which reduces to the form

$$(\lambda^2-3)(\lambda \sin A \sin B \sin C+1+\cos A \cos B \cos C)=0.$$

The coordinates of the point circle, for which $\lambda=\sqrt{3}$, must be given by substituting this value in the equations of the centre of a circle (9).

Then
$$\begin{aligned}
\alpha &= R(\sin A+\sqrt{3}\cos A)/(\sqrt{3}+\cot\omega)\\
&= 2R\cos(A-60^\circ)/(\sqrt{3}+\cot\omega),\\
\beta &= 2R\cos(B-60^\circ)/(\sqrt{3}+\cot\omega),\\
\gamma &= 2R(\cos C-60^\circ)/(\sqrt{3}+\cot\omega),
\end{aligned}$$

and the remaining value

$$\lambda \sin A \sin B \sin C+1+\cos A\cos B\cos C=0,$$

or
$$\lambda=-\cot\omega,$$

gives all the coordinates of the centre infinite. This shows that the circle is a straight line which is at once found to be

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=0.$$

The centre of the circle

$$\Sigma(\beta\cos B-\gamma\cos C)^2+\Sigma(\beta\sin B+\gamma\sin C)^2\equiv\Sigma\alpha^2+\Sigma\beta\gamma\cos A=0$$

is Lemoine's point K, and its radius is

$$R\tan\omega\sqrt{(-3)}.$$

The equation of the real circle corresponding to this (centre K radius $=R\tan\omega\sqrt{3}$) is

$$4\Delta\left\{\Sigma\alpha^2+\Sigma\beta\gamma\cos A\right\}-3\tan\omega\left\{\Sigma a\alpha\right\}^2=0.$$

We will now for a time desert analysis and employ inversion, using inversion in the sense that we take a fixed point O_n, and find a point Q corresponding to any other point P, such that O_n, P, Q are in a straight line, and the rectangle O_nP, O_nQ is equal to a constant (the square of the radius of inversion).

If we take the centre and radius of the circle ABC as the centre and radius of inversion, the three-ring circles invert into themselves, and a circle cutting them at the angle θ must invert into a circle cutting them at the angle θ; we see, therefore, that the $+\lambda$ circle of Schoute's series inverts into the $-\lambda$ circle of his series, and if V, W be the isodynamic points, $OV.OW=R^2$. It also follows that O

is the external centre of similitude of the $\pm\lambda$ pair of Schoute's circles.

If, again, we take O_n the centre of inversion on Lemoine's line—the radical axis of the system—and the tangent from O_n to the circle ABC as the radius of inversion, each circle of Schoute's system will invert into itself; but the three-ring circles will assume a different position for each position of O_n, and will always possess Schoute's system of circles *each for each* as before. Their points of contact will accordingly form new triangles, each of which possesses the same system of Schoute's circles. That these are the co-Brocardal triangles of ABC can be proved by finding the envelope of a side as O_n moves along Lemoine's line, or thus:—The three-ring circles of the co-Brocardal triangles must touch a pair of circles, which are coaxal with the Brocard circle and the circle ABC, and must be orthogonal to circle ABC; and the only rings of three circles that satisfy these conditions are the rings of circles obtained by our inversion.

This can also be proved thus. All our three-ring circles touch or cut all Schoute's circles at the same angle. So, taking (l, m, n), $(\sec A, 0, 0)$ as two specimens of three-ring circles, we have, by (11),

$$\Sigma l \sin A = \sqrt{\{\Sigma l^2 \sin^2 A - 2\Sigma mn \sin B \sin C\}},$$

or $$\Sigma mn \sin B \sin C = 0,$$

and any ring-circle has an equation of the form

$$l \sin A \,.\, A_1 + m \sin B \,.\, B_1 + n \sin C \,.\, C_1 = 0.$$

This gives, as the form of the general equation of a three-ring circle,

$$A_2 \equiv A_1 - B_1 (1+\mu) - C_1 \left(1 + \frac{1}{\mu}\right) = 0.$$

The common chord of this circle and the circle ABC is

$$\mu \frac{\alpha}{a} - \mu (1+\mu) \frac{\beta}{b} - (1+\mu) \frac{\gamma}{c} = 0,$$

and, as this is a quadratic equation in μ, the envelope of all such lines is given by the equation

$$\left(\frac{\alpha}{a} - \frac{\beta}{b} - \frac{\gamma}{c}\right)^2 = 4 \frac{\beta\gamma}{bc},$$

which is the equation of Brocard's ellipse.

The circle A_2 meets the circle ABC where

$$A_1 B_1 + B_1 C_1 + C_1 A_1 \equiv \{\Sigma \beta\gamma \sin A\}^2 = 0,$$

and $$A_1 - B_1(1+\mu) - C_1\left(1+\frac{1}{\mu}\right) = 0;$$

that is, where $$\mu A_1 = -(\mu+1)\,B_1 = C_1,$$

and where $$A_1 = \mu B_1 = -(\mu+1)\,C_1.$$

The third point of this co-Brocardal triangle* must be

$$-(\mu+1)\,A_1 = B_1 = \mu C_1,$$

and the equation to the other circles of the same ring will be

$$B_2 \equiv -A_1\left(1+\frac{1}{\mu}\right) + B_1 - C_1(1+\mu) = 0,$$

$$C_2 \equiv -A_1(1+\mu) - B_1\left(1+\frac{1}{\mu}\right) + C_1 = 0.$$

Here A_2, B_2, C_2 satisfy the relations

$$A_2 + B_2 + C_2 \equiv -\left(\mu+1+\frac{1}{\mu}\right)(A_1+B_1+C_1),$$

$$B_2C_2 + C_2A_2 + A_2B_2 \equiv \left(\mu+1+\frac{1}{\mu}\right)^2 (B_1C_1 + C_1A_1 + A_1B_1).$$

Again, we will invert with regard to one of the isodynamic points V, W as our centre of inversion, and the tangent to the circle ABC as our radius of inversion. Since

$$OV.OW = R^2,$$

$$WV.WO = WO^2 - R^2 = WT^2;$$

therefore, inverting with respect to W, as above, V inverts into the centre of the circle ABC, and the circle ABC inverts into itself; and two circles of the *coaxal* system having become concentric by inversion, the rest must have done so also, and the whole system of Schoute's circles becomes a concentric system, and, in consequence of these circles having become *concentric*, the three-ring circles become necessarily equal circles, and their points of contact form an equi-

* The coordinates of the angular points of this triangle are given by the equations

$$\frac{\alpha}{a} = \mu\frac{\beta}{b} = -(\mu+1)\frac{\gamma}{c},$$

$$-(\mu+1)\frac{\alpha}{a} = \frac{\beta}{b} = \mu\frac{\gamma}{c},$$

$$\mu\frac{\alpha}{a} = -(\mu+1)\frac{\beta}{b} = \frac{\gamma}{c}.$$

lateral triangle. This applies not only to our original triangle, but to any of the co-Brocardal system of triangles; and as at the same time the other isodynamic point inverts into the centre of the circle ABC, the circles of Apollonius, which passed through both points V, W and the angular points of the triangle, become straight lines passing through O and the angular points of the equilateral triangles, that is to say, become diameters of the circle ABC through the angular points of the equilateral triangles. Therefore in any triangle the circles of Apollonius cut one another at an angle of 60°.

The circle of inversion in this case is at the isodynamic point—in other words, is concentric with the circle

$$\alpha^2+\beta^2+\gamma^2+\Sigma\beta\gamma\,(\cos A\pm\sqrt{3}\sin A)=0.$$

[The $-$ sign gives the inner point (V), the $+$ sign the outer point (W)]. A circle concentric with this must have an equation of the form

$$\Sigma\alpha^2+\Sigma\beta\gamma\,(\cos A\pm\sqrt{3}\sin A)+h\,(\Sigma\alpha\sin A)^2=0.$$

If this be also of the form (4),

$$l\sin A=1+h\sin^2 A,$$

$$m\sin B=1+h\sin^2 B,$$

$$n\sin C=1+h\sin^2 C,$$

$$-l\sin A\cos A+m\sin B\cos A+n\sin C\cos A$$

$$=\cos A\pm\sqrt{3}\sin A+2h\sin B\sin C,$$

and also from above $\quad = \cos A\left[1+h\,(-\sin^2 A+\sin^2 B+\sin^2 C)\right]$;

whence

$$h\left[\cos A\,(\sin^2 A-\sin^2 B-\sin^2 C)+2\sin B\sin C\right]=\mp\sqrt{3}\sin A,$$

or $$h=\frac{\mp\sqrt{3}}{2\sin A\sin B\sin C},$$

and the equation of the required circle becomes

$$\Sigma\alpha^2+\Sigma\beta\gamma\,(\cos A\pm\sqrt{3}\sin A)\mp\frac{\sqrt{3}}{2\sin A\sin B\sin C}(\Sigma\alpha\sin A)^2=0.$$

The upper signs will give an imaginary and the lower a real circle,

showing that the inversion is in the first case across the point, in the second away from it.

The centre of the circle

$$\alpha^2+\beta^2+\gamma^2+\Sigma\beta\gamma\cos A = 0$$

was found to be the symmedian point K_1, and its radius

$$\rho = R\tan\omega\,\sqrt{(-3)}.$$

Again, if we draw the chord AKA' of the circle ABC, the product $AK.KA'$ is equal to

$$R^2-OK^2 = R^2-4\rho'^2$$

(where ρ' is the radius of the Brocard circle, *i.e.*, of the Schoute circle for which $\lambda = \cot\omega$) $= R^2-R^2(1-3\tan^2\omega)$, by (10),

$$= 3R^2\tan^2\omega.$$

This proves that, if we invert with K as centre of inversion, and the radius of this impossible circle as the radius of inversion, the circle ABC will invert into itself; consequently, all circles orthogonal to it will invert into circles orthogonal to it, and the three-ring circles of the triangle ABC will invert into the three-ring circles of the co-symmedian triangle; as this is a co-Brocardal triangle, the system of Schoute's circles must invert into one another as circles cut one another at the same angles as their inverses. It also follows that K is the internal centre of similitude of each pair $(\pm\lambda)$ of Schoute's circles.

We can by means of this result obtain the equations of the six-ring circles which are orthogonal to the circle ABC, and are cut at equal angles by each of the Schoute circles. For, drawing the chords AKA', BKB', CKC' of the circle ABC, A', B', C' must, by our last result, be the points where the six-ring circles at ABC meet the circle ABC again; and as the coordinates of K are $a : b : c$, the equation of $A'K$ is

$$\beta\sin C = \gamma\sin B,$$

and the equations of $A'B$, $A'C$ are found by eliminating β and γ from the equations of $A'K$ and the circle ABC; therefore the equation of $A'B$ is

$$2\alpha\sin C+\gamma\sin A = 0\,;$$

and therefore, by equation (4), a ring-circle of the six-ring through $A'B$ will have for its equation

$$2A_1+C_1 = 0,$$

and the equations of a complete ring of the six-ring circles will be

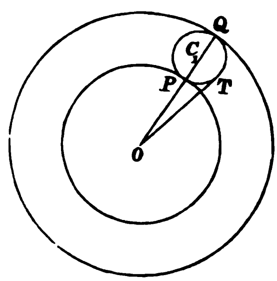
$$B_1+2C_1 = 0,$$
$$A_1+2C_1 = 0,$$
$$C_1+2A_1 = 0,$$
$$B_1+2A_1 = 0,$$
$$A_1+2B_1 = 0,$$
$$C_1+2B_1 = 0.$$

The value of λ for the Schoute circles that touch the six-ri circles is

$$\lambda = \pm 2\sqrt{3}.$$

The general equation of a member of the six-ring circles can be fou by taking an arbitrary point O_n on Lemoine's line, drawing the lin O_nA, O_nB', and finding where these meet the circle ABC again, a forming the equation of that chord of the circle ABC; whereupon at once know the equation of the corresponding ring-circle, and pr ceeding in the same way we can find the equations of all the circl of any six-ring system.

We will briefly indicate how the same can be done for any oth ring. The chords common to the circles of the ring and the circur scribed circle form a harmonic polygon (Casey's "Sequel to Euclid 199–206), and always touch an ellipse of a family of which Brocard is the best known example. Their centres lie on another ellipse who foci are the centres of a $\pm\lambda$ pair of Schoute's circles.



To determine the value of λ for the tangent circle to any rin formula (11) shows us that the angle at which any of Schoute's circl cuts the three-ring circles depends only on the value of λ, and not on th angles A, B, C; so we can determine the value of λ for an equilater triangle.

Now, for any ring of r circles, we must have in the above figure—where C_1 is the centre of PTQ one of the ring-circles, and P, T, Q are the points where the ring-circle meets the λ-Schoute circle, the circle ABC, and the $-\lambda$-Schoute circle—

$$\tan\frac{\pi}{r} = \tan C_1OT = C_1T/\sqrt{(OP.\,OQ)} = \tfrac{1}{2}(OQ-OP)/\sqrt{(OP.\,OQ)},$$

and therefore, taking the values of OP, OQ from equation (10) with due regard to sign, and remembering that $\lambda < \sqrt{3}$ numerically, and

$$\cot\omega = \sqrt{3},$$

we have $$\tan\frac{\pi}{r} = \frac{1}{2}\left(\frac{1}{\lambda-\cot\omega}-\frac{1}{\lambda+\cot\omega}\right)\bigg/\sqrt{\left(\frac{1}{\lambda^2-\cot^2\omega}\right)}$$

$$= \frac{\cot\omega}{\sqrt{(\lambda^2-\cot^2\omega)}} = \frac{\sqrt{3}}{\sqrt{(\lambda^2-3)}},$$

$$\lambda^2 = 3\operatorname{cosec}^2\frac{\pi}{r},$$

$$\lambda = \pm\sqrt{3}\operatorname{cosec}\frac{\pi}{r},$$

and for the six-ring circle

$$\lambda = \pm 2\sqrt{3}.$$

The general relation between l, m, n, when a circle belongs to the ring of r circles, is found by substituting the values

$$\cos\theta = 1 \quad\text{and}\quad \lambda = \pm\sqrt{3}\operatorname{cosec}\frac{\pi}{r}$$

from above in formula (11), and can be written

$$(\Sigma l\sin A)\tan\frac{\pi}{r} = \sqrt{3}\,.\,\sqrt{(\Sigma l^2\sin^2 A - 2\Sigma mn\sin B\sin C)} \quad\ldots(12)$$

and this is, consequently, also the relation between l, m, n when

$$l\alpha+m\beta+n\gamma = 0$$

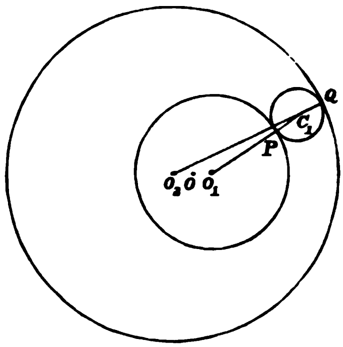
is a side of an harmonic polygon of r sides belonging to the same system.

The locus of the centres of the ring-circles of any series is plainly an ellipse or other conic of which the foci are the centres of the $\pm\lambda$ circles that touch all the members of that ring.

For, if O_1, O_2 be the centres of the $\pm\lambda$ circles that touch at P and

Q the ring-circle whose centre is C_1, then

$$O_1C_1 + C_1O_2 = R_1 - C_1P + R_2 + C_1Q = R_1 + R_2$$
$$= \text{a constant} = 2\lambda R\sqrt{(\lambda^2-3)}/(\lambda^2-\cot^2\omega).$$



However, we can find the equation of this locus more readily by finding the locus of the pole of

$$l\alpha + m\beta + n\gamma = 0,$$

subject to the condition (12), which may be written in the form

$$(\Sigma l \sin A)^2 \left(\tan^2 \frac{\pi}{r} + 3\right) = 6\Sigma l^2 \sin^2 A.$$

Since α, β, γ, a point on the locus, is the pole of

$$l\alpha + m\beta + n\gamma = 0$$

with respect to the circle ABC,

$$\frac{l}{b\gamma + c\beta} = \frac{m}{a\gamma + c\alpha} = \frac{n}{a\beta + b\alpha},$$

and the locus of the pole must therefore be given by the equation

$$\{\Sigma (ab\gamma + ac\beta)\}^2 \left(\tan^2 \frac{\pi}{r} + 3\right) = 6\Sigma (ab\gamma + ac\beta)^2,$$

$$(2\Sigma bca)^2 \left(\tan^2 \frac{\pi}{r} + 3\right) = 12\Sigma (b^2c^2\alpha^2 + a^2bc\beta\gamma),$$

$$\Sigma (b^2c^2\alpha^2 + 2a^2bc\beta\gamma) \left(\tan^2 \frac{\pi}{r} + 3\right) = 3\Sigma (b^2c^2\alpha^2 + a^2bc\beta\gamma),$$

$$\Sigma b^2c^2\alpha^2 \tan^2 \frac{\pi}{r} + \Sigma a^2bc\beta\gamma \left(2\tan^2 \frac{\pi}{r} + 3\right) = 0.$$

It is plain that the sides also of the harmonic polygons, being the polars of these centres with respect to the circle ABC, must envelope an ellipse.

Its equation, being the envelope of

$$l\alpha+m\beta+n\gamma=0$$

subject to the condition (12), is

$$\Sigma\, b^2c^2\alpha^2 \tan^2\frac{\pi}{r}-\left(\tan^2\frac{\pi}{r}+3\right)\Sigma a^2bc\beta\gamma=0.$$

Both series of ellipses belong to the same family

$$\Sigma\, b^2c^2\alpha^2+\mu\Sigma\, a^2bc\beta\gamma=0 \qquad\qquad (13),$$

and the values of μ for a pair of them are connected by the relation

$$\mu_1+\mu_2=1.$$

The usual formulæ for the foci of the conic

$$A\alpha^2+B\beta^2+C\gamma^2+2D\beta\gamma+2E\gamma\alpha+2F\alpha\beta=0$$

are $\quad \alpha^2\,(A'a^2+B'b^2+C'c^2+2D'bc+2E'ca+2F'ab)$

$$-4\Delta\alpha\,(bF'+cE'+aA')+4\Delta^2A',$$

$=$ two like expressions in β and γ, where A', B', &c. are the minors of A, B, &c. in the determinant

$$\begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix} \quad \text{(Whitworth, p. 269).}$$

In the present case these equations reduce to

$$\alpha^2\left\{\mu\,(-\Sigma a^4+2\Sigma b^2c^2)-2\Sigma a^4\right\}$$
$$-4\Delta a\alpha\left\{(b^2+c^2-a^2)\,\mu-2a^2\right\}-4\Delta^2a^2\,(\mu+2)=\&c.$$

Putting each of these $=K$, and rearranging,

$$\alpha^2\left\{\mu\,(\Sigma a^2)^2-2\,(\mu+1)\,\Sigma a^4\right\}-4\Delta a\alpha\,\left\{\mu\Sigma a^2-2\,(\mu+1)\,a^2\right\}$$
$$+4\Delta^2a^2\left\{\mu-2\,(\mu+1)\right\}=K,$$

and two like equations.

Eliminating from these the ratios

$$\mu:-2\,(\mu+1):K,$$

we obtain, as the equation of the locus of the foci,

$$\begin{vmatrix} (\alpha\Sigma a^2-2\Delta a)^2, & \alpha^2\Sigma a^4-4\Delta a^3\alpha+4\Delta^2a^2, & 1 \\ (\beta\Sigma a^2-2\Delta b)^2, & \beta^2\Sigma a^4-4\Delta b^3\beta+4\Delta^2b^2, & 1 \\ (\gamma\Sigma a^2-2\Delta c)^2, & \gamma^2\Sigma a^4-4\Delta c^3\gamma+4\Delta^2c^2, & 1 \end{vmatrix} = 0,$$

which, on expansion, yields the two factors

$$\Sigma bc\,(b^2-c^2)\,\alpha \quad \text{and} \quad abc\Sigma a^2-\Sigma a^3\beta\gamma.$$

For the locus the real foci are on the straight line KO,

$$\Sigma bc\,(b^2-c^2)\,\alpha = 0,$$

and the imaginary ones on the Brocard circle

$$abc\Sigma a^2 = \Sigma a^3\beta\gamma.$$

Since the equation (13) can be arranged in the form

$$(\Sigma bc\alpha)^2+\overline{\mu-2}\,\Sigma a^2bc\beta\gamma = 0,$$

the family of ellipses (13) have imaginary double contact with one another and the circle ABC where they meet the line

$$\Sigma bc\alpha = 0.$$

There is one parabola belonging to the series, which is the locus of centres of circles that touch Brocard's circle and Lemoine's line, and then $$\mu = \tan^2\omega-3.$$

(Lemoine's line)^2 is given twice over by making the discriminant vanish, and we also obtain Lemoine's point as a particular case of these conics.

To find the general equation of a circle of Apollonius, we know that it is orthogonal to the circle ABC, and therefore its equation is of the form (4),

$$l\sin A\,.\,A_1+m\sin B\,.\,B_1+n\sin C\,.\,C_1 = 0.$$

To be a circle of Apollonius it must also pass through the point

$$\sin(A+60^\circ) : \sin(B+60^\circ) : \sin(C+60^\circ).$$

Making these substitutions for α, β, γ in A_1, we obtain, for

$$\alpha^2-\beta\gamma\cos A+\gamma\alpha\cos B+\alpha\beta\cos C,$$

$$\begin{aligned} \tfrac{1}{4}\big\{\sin^2 A+3\cos^2 A-\cos A\,(\sin B\sin C+3\cos B\cos C) \\ +\cos B\,(\sin A\sin C+3\cos A\cos C) \\ +\cos C\,(\sin A\sin B+3\cos A\cos B)\big\} \\ +\tfrac{1}{4}\sqrt{3}\,\big\{2\sin A\cos A-\sin A\cos A+\sin B\cos B+\sin C\cos C\big\} \\ = \tfrac{1}{2}\big\{1+\cos A\cos B\cos C+\sqrt{3}\sin A\sin B\sin C\big\}\,; \end{aligned}$$

therefore, in this case, $A_1 = B_1 = C_1$,

and the necessary condition for the above equation (4) to represent a circle of Apollonius is

$$l \sin A + m \sin B + n \sin C = 0.$$

The particular one through the angular point A of the original triangle must satisfy (4), when we put

$$\alpha : \beta : \gamma = 1 : 0 : 0;$$

A_1 becomes 1, $B_1 = C_1 = 0$,

therefore $l = 0$,

and the equation reduces to $B_1 = C_1$.

Hence $B_1 = C_1, \quad C_1 = A_1, \quad A_1 = B_1$

are the three primary circles of Apollonius. Any other can be found by making (4) pass through some other point on the circle ABC. The circle on VW as diameter is plainly the smallest of all these circles, and its equation can be found by making its centre lie on OK whose equation is

$$\Sigma a (b^2 - c^2)\, \alpha = 0,$$

or, more simply, by making the common chord with the circle ABC,

$$l\alpha + m\beta + n\gamma = 0,$$

parallel to Lemoine's line. The condition for this is

$$\begin{vmatrix} l, & m, & n \\ \frac{1}{a}, & \frac{1}{b}, & \frac{1}{c} \\ a, & b, & c \end{vmatrix} = 0,$$

or $$la (b^2 - c^2) + mb (c^2 - a^2) + nc (a^2 - b^2) = 0.$$

And the equation to this circle may be obtained by eliminating l, m, n from the three equations

$$l \sin A \,.\, A_1 + m \sin B \,.\, B_1 + n \sin C \,.\, C_1 = 0,$$

$$l \sin A \quad + m \sin B \quad + n \sin C \quad = 0,$$

$$l \sin A (\sin^2 B - \sin^2 C) + m \sin B (\sin^2 C - \sin^2 A)$$
$$+ n \sin C (\sin^2 A - \sin^2 B) = 0.$$

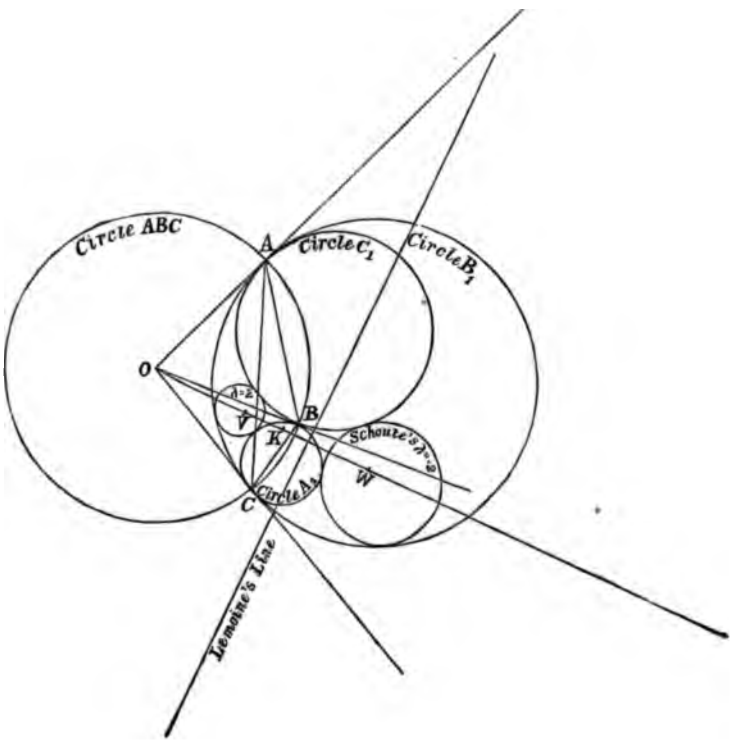
Its equation, therefore, is

$$\begin{vmatrix} A_1, & B_1, & C_1 \\ 1, & 1, & 1 \\ \sin^2 B - \sin^2 C, & \sin^2 C - \sin^2 A, & \sin^2 A - \sin^2 B \end{vmatrix} = 0,$$

or $$A_1\,(2a^2 - b^2 - c^2) + B_1\,(2b^2 - a^2 - c^2) + C_1\,(2c^2 - a^2 - b^2) = 0.$$

We have previously omitted to remark that, if we take any other point anywhere, and invert with the tangent to the circle ABC as the radius of inversion, we shall obtain a new set of harmonic polygons, ring-circles and Schoute's circles; as is evident, since all curves cut at the same angle as their inverses.

It is worthy of notice that the circles ABC, BCW, CAW, and ABW, being the inverses of the circle ABC, and of the sides of an equilateral triangle, intersect at angles of 60°, and the circles of Apollonius round VAW, VBW, VCW, being the inverses of the bisectors of the angles of the equilateral triangle, intersect at 60° and bisect the angles between the circles BCW, CAW, ABW.



Presents received during the Recess :—

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"The Scientific Proceedings of the Royal Dublin Society," Vol. VI., Parts III. to VI.

"Memoirs of the National Academy of Sciences," Vol. IV., Part I.; Washington, 1888.

"Annals of Mathematics," Vol. IV., No. 6; Vol. V., No. 1; Virginia, 1889.

"Bulletin des Sciences Mathématiques," Tome XIII., June to September, 1889.

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"Jornal des Sciencias Mathematicas e Astronomicas," Vol. IX., No. 2.

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo II., Nos. 8 to 10.

"Rendiconti del Circolo Matematico di Palermo," Fasc. III.—V.

"The Mathematical Theory of Electricity and Magnetism," by H. W. Watson, Sc.D., F.R.S., and S. H. Burbury, M.A., Vol. II. "Magnetism and Electrodynamics," Oxford, Clarendon Press, 1889.

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"Smithsonian Report," 1886, Part I., 8vo; Washington, 1889.

"Acta Mathematica," XII., 3 and 4.

"Annali di Matematica," Tomo XVII., Fasc. 2.

"Œuvres complètes de Christiaan Huygens," Tome II., 4to; La Haye, 1889.

"Bulletins de l'Académie Royale de Belgique," 3me Serie, T. XIV. to XVII., 1887, 88, 89; Annuaire, 1888, 1889.

"Journal für die reine und angewandte Mathematik," Band 105, Hefte I. and II.

"Journal de l'École Polytechnique," Cahier 58; Paris, 1889.

"Sitzungsberichte der Physikalisch-medicinischen Societät in Erlangen," 1888; Munchen, 1889.

"Journal of the College of Science, Imperial University, Japan," Vol. III., Parts I. and II.

Pamphlets by M. Maurice d'Ocagne:—

"Sur certaines Courbes qu'on peut adjoindre aux Courbes Planes pour l'étude de leurs Propriétés Infinitésimales." (*American Jour. of Math.*, Vol. XI., No. 1.)

"Calcul direct des Termes d'une réduite de rang quelconque d'une Fraction Continue Périodique."

"Détermination du Rayon de Courbure de la Courbe Intégrale." (*Nouvelles Annales.*)

"Quelques propriétés de l'Ellipse; Deviation, Ecart normal." (*Nouvelles Annales.*)

"Sur les Systèmes de Péninvariants principaux d'une Forme Binaire. (*Bulletin de la S. Math. de France.*)

"Formules nouvelles pour résoudre le problème de la Carte au moyen de données particulières." (*Revue Maritime et Coloniale*, Fev., 1889.)

APPENDIX.

Mr. Basset points out that the following corrections should be supplied in his investigation of the stability of Maclaurin's Spheroid (Vol. XIX., pp. 52–54):—"The correct result, which was first obtained by Riemann, is that for an ellipsoidal displacement, the spheroid becomes unstable, when the excentricity exceeds the root of the equation

$$e\,(1-e^2)^{\frac{1}{2}}\,(3+4e^2) = (3+2e^2-4e^4)\sin^{-1}e,$$

which gives e equal to about ·95." See his work on "Hydrodynamics," Vol. II., p. 124.

The following recently published papers bear upon the subject:—

Love, *Phil. Mag.*, Vol. XXVII., p. 254.

Bryan, *Phil. Trans.*, 1889, p. 187, and *Proc. Camb. Phil. Soc.*, Vol. VI., p. 248.

There is a paper by H. Weber in the *Math. Annalen*, Band XXXIII., Heft. 3, p. 391, "Zur complexen Multiplication elliptischen Funktionen." (See Prof. Greenhill's paper on "Complex Multiplication Moduli of Elliptic Functions," Vol. XIX., p. 362.)

The following is a statement of results arrived at by Dr. Wolstenholme in connexion with his communication, "Certain Algebraical results deduced from the Geometry of the Quadrangle and Tetrahedron" (p. 3) :—

Properties of Semi-equi-facial Tetrahedra (in which the sum of the areas of two faces is equal to the sum of the areas of the other two faces.)

1. NOTATION.—In a tetrahedron $OABC$, the plane angles at O are $\alpha_0\beta_0\gamma_0$; at A, $\alpha_1\beta_1\gamma_1$; at B, $\alpha_2\beta_2\gamma_2$; and at C, $\alpha_3\beta_3\gamma_3$; each angle α being opposite to BC or OA, each angle β to CA or OB, and each angle γ to AB or OC. The lengths of the edges OA, OB, OC are a, b, c; those of BC, CA, AB are x, y, z. The areas of the faces opposite O, A, B, C are Δ_0, Δ_1, Δ_2, Δ_3. The dihedral angles are denoted by A, B, C, X, Y, Z, corresponding to the edges to which they are opposite; and V denotes the volume of the tetrahedron.

2. Now suppose that, in a real finite tetrahedron, $\Delta_0+\Delta_1 = \Delta_2+\Delta_3$. Then I assert that all the following relations are also true :—

$$\left.\begin{aligned} \alpha_0+\alpha_1 &= \alpha_2+\alpha_3 \\ \beta_0+\beta_1 &= \beta_2+\beta_3 \\ \gamma_0+\gamma_1 &= \gamma_2+\gamma_3 \end{aligned}\right\}\ldots\ldots(\text{A}),\qquad \left.\begin{aligned} \alpha_0+\gamma_0-\beta_0 &= \alpha_3+\gamma_3-\beta_3 \\ \alpha_0+\beta_0-\gamma_0 &= \alpha_2+\beta_2-\gamma_2 \\ \alpha_1+\beta_1-\gamma_1 &= \alpha_3+\beta_3-\gamma_3 \\ \alpha_1+\gamma_1-\beta_1 &= \alpha_2+\gamma_2-\beta_2 \end{aligned}\right\}\ldots\ldots(\text{B}),$$

$$\frac{a+x}{\sin\frac{1}{2}(A+X)} = \frac{x-a}{\sin\frac{1}{2}(A-X)} = \frac{b+y}{\sin\frac{1}{2}(B+Y)} = \frac{b-y}{\sin\frac{1}{2}(B-Y)}$$

$$= \frac{c+z}{\sin\frac{1}{2}(C+Z)} = \frac{c-z}{\sin\frac{1}{2}(C-Z)} = \frac{4}{3}\,\frac{\sqrt{\Delta_0\Delta_1\Delta_2\Delta_3}}{V}\ldots\ldots(\text{C}).$$

Of the equations (A) I can prove the first without difficulty, but have not yet succeeded in proving the second and third *each* to hold, although, from the number of numerical cases I have tried, taking the values of a, b, c, y, z quite at random, and then determining x so as to satisfy the equation $\Delta_0+\Delta_1 = \Delta_2+\Delta_3$, and finding that in every case these equations are accurately correct, I can have no doubt whatever that the three equations (A) do hold. The system (B) is very easily deduced from (A), and the system

$$\alpha_1+\beta_2+\gamma_3 = \pi,\quad \alpha_0+\beta_3+\gamma_2 = \pi,\quad \alpha_3+\beta_0+\gamma_1 = \pi,\quad \alpha_2+\beta_1+\gamma_0 = \pi.$$

Equation (C) I have, as yet, deduced only from observation of calculated tetrahedra of this kind $\Big($except the last member $\frac{4}{3}\,\frac{\sqrt{\Delta_0\Delta_1\Delta_2\Delta_3}}{V}$, which

I have deduced by simple mathematics from the others, assumed to be equal). I do not at present see any promising mode of attacking these, as all the expressions I can find involve the *surds* Δ_0, Δ_1, Δ_2, Δ_3. This system may also be written

$$\frac{a^2-x^2}{\cos A-\cos X}=\frac{a^2+x^2}{1-\cos A\cos X}=\frac{2ax}{\sin A\sin X}=\&c.,$$

and since $\sin A=\dfrac{3Vx}{2\Delta_0\Delta_1}$,

$$\sin X=\frac{3Va}{2\Delta_2\Delta_3},\qquad \frac{ax}{\sin A\sin X}=\frac{4}{9}\,\frac{\Delta_0\Delta_1\Delta_2\Delta_3}{V^2}.$$

3. More generally, I find that the sign of

$$\Delta_0+\Delta_1-\Delta_2-\Delta_3$$

is, in any real finite tetrahedron, also the sign of

$$\alpha_0+\alpha_1-\alpha_2-\alpha_3,\quad \beta_0+\beta_1-\beta_2-\beta_3,\quad \gamma_0+\gamma_1-\gamma_2-\gamma_3;$$

and of
$$(\alpha_0+\gamma_0-\beta_0)-(\alpha_3+\gamma_3-\beta_3),$$

and the other expression of (B) which vanish when

$$\Delta_0+\Delta_1=\Delta_2+\Delta_3.$$

These results are not all true for a flat tetrahedron of any kind.

4. The four expressions Δ_0, Δ_1, Δ_2, Δ_3 are all irrational expressions of the form
$$\sqrt{s(s-\alpha)(s-\beta)(s-\gamma)},$$

where α is always either a or x, β always b or y, and γ always either c or z.

Let
$$\Delta,\ \Delta_1',\ \Delta_2',\ \Delta_3'$$

denote the results of interchanging the β, γ factors of Δ_0, Δ_1, and also of Δ_2, Δ_3; then
$$\Delta_0'+\Delta_1'-\Delta_2'-\Delta_3'$$

is always of the same sign as

$$\Delta_0+\Delta_1-\Delta_2-\Delta_3.$$

Let
$$\Delta_0^2,\ \Delta_1^2,\ \Delta_2^2,\ \Delta_3^2$$

denote the results of interchanging the β, γ factors in Δ_0, Δ_2, and also in Δ_1, Δ_3; then

$$b\Delta_0^2+y\Delta_1^2-b\Delta_2^2-y\Delta_3^2$$

is always of the sign *opposite* to that of $\Delta_0+\Delta_1-\Delta_2-\Delta_3$.

Next, let $$\Delta_0^3,\ \Delta_1^3,\ \Delta_2^3,\ \Delta_3^3$$

denote the results of interchanging the β, γ factors in Δ_0, Δ_3, and also in Δ_1, Δ_2; then

$$c\Delta_0^3+z\Delta_1^3-z\Delta_2^3-c\Delta_3^3$$

is of the *opposite* sign to that of $\Delta_0+\Delta_1-\Delta_2-\Delta_3$. Interchanging the γ, a factors, and the a, β factors in the same way and in the same order, we get six more sets, and

$$a\Delta_0^4+a\Delta_1^4-x\Delta_2^4-x\Delta_3^4 \text{ is of } \textit{opposite},$$

$$\Delta_0^5+\Delta_1^5-\Delta_2^5-\Delta_3^5,\quad c\Delta_0^6+z\Delta_1^6-z\Delta_2^6-c\Delta_3^6$$

are of the *same* sign as $\Delta_0+\Delta_1-\Delta_2-\Delta_3$. So

$$a\Delta_0^7+a\Delta_1^7-x\Delta_2^7-x\Delta_3^7 \text{ is of opposite sign,}$$

$$b\Delta_0^8+y\Delta_1^8-b\Delta_2^8-y\Delta_3^8,\quad \Delta_0^9+\Delta_1^9-\Delta_2^9-\Delta_3^9$$

of the same sign as $\Delta_0+\Delta_1-\Delta_2-\Delta_3$.

These results are not true for any flat tetrahedron, but corresponding results, which may easily be obtained, are. I ought to except the first $\Delta_0'+\Delta_1'-\Delta_2'-\Delta_3'$, which has always the same sign as $\Delta_0+\Delta_1-\Delta_2-\Delta_3$, whether the volume of the tetrahedron be finite or zero.

5. The results stated in (4) are all easily proved, assuming those stated in (3). The first of these, that $a_0+a_1-a_2-a_3$ is always of the same sign as $\Delta_0+\Delta_1-\Delta_2-\Delta_3$, I can prove without difficulty; and we hence deduce the algebraical theorem, which is of a quite novel and surprising form to me. Denoting

$$x+y+z\equiv 2s_0,\quad x+b+z\equiv 2s_1,\quad a+y+c\equiv 2s_2,\quad a+b+z\equiv 2s_3,$$

the two irrational expressions

$$\sqrt{s_0\,.\,s_0-x\,.\,s_0-y\,.\,s_0-z}+\sqrt{s_1\,.\,s_1-x\,.\,s_1-b\,.\,s_1-c}$$
$$-\sqrt{s_2\,.\,s_2-a\,.\,s_2-y\,.\,s_2-c}-\sqrt{s_3\,.\,s_3-a\,.\,s_3-b\,.\,s_3-z}$$

and

$$\sqrt{s_0\,.\,s_0-x\,.\,s_1-b\,.\,s_1-c}+\sqrt{s_1\,.\,s_1-x\,.\,s_0-y\,.\,s_0-z}$$
$$-\sqrt{s_2\,.\,s_2-a\,.\,s_3-b\,.\,s_3-z}-\sqrt{s_3\,.\,s_3-a\,.\,s_2-y\,.\,s_2-c}$$

have always the same sign, for any values of a, b, c, x, y, z for which the surds are real. Whether the same holds between impossibles, I have not at all tried to discover. In the above, the sign of the square root $\sqrt{\ }$ implies always the arithmetical square root.

The note by Mr. H. M. Taylor (p. 88) was entitled "On the Developable Surface through two Conics inscribed (or escribed) in two of the faces of a Tetrahedron." Mr. Taylor assumes that the surface passes through the two conics whose equations are (in tetrahedral coordinates with the given tetrahedron as the tetrahedron of reference)

$$w = 0, \quad \sqrt{ax} + \sqrt{by} + \sqrt{cz} = 0 \quad \text{.....................(1)},$$

and

$$x = 0, \quad \sqrt{\beta y} + \sqrt{\gamma z} + \sqrt{\delta w} = 0 \quad \text{.....................(2)}.$$

The problem of finding the equation of the developable surface is the same as that of finding the envelope of the plane

$$Lx + My + Nz + Pw = 0 \quad \text{.......................(i.)},$$

under the conditions

$$a/L + b/M + c/N = 0 \quad \text{.......................(ii.)},$$

$$\beta/M + \gamma/N + \delta/P = 0 \quad \text{.......................(iii.)}.$$

Eliminating $1/L$ and $1/P$ from these, he gets a result of the form

$$A\left(\frac{M}{N}\right)^3 + B\left(\frac{M}{N}\right)^2 + C\left(\frac{M}{N}\right) + D = 0 \quad \text{............(iv.)},$$

where

$$\left.\begin{aligned} A &= c\gamma y, \quad D = b\beta z, \\ B &= -a\gamma x + (b\gamma + \beta c)\,y + c\gamma z - c\delta w \\ C &= -a\beta x + b\beta y + (b\gamma + c\beta)\,z - b\delta w \end{aligned}\right\} \quad \text{............(3)}.$$

The equation of the required envelope is the discriminant of (iv.) considered as an equation in M/N. It is

$$27A^2D^2 - B^2C^2 + 4AC^3 + 4B^3D - 18ABCD = 0 \quad \text{...........(v.)},$$

an equation of the 4th degree in x, y, z, w.

If we consider the section of the surface by the plane $y = 0$; then $A = 0$, and (v.) becomes

$$B^2(C^2 - 4BD) = 0.$$

Now $B = 0$ gives

$$-a\gamma x + c\gamma z - c\delta w = 0 \quad \text{.......................(vi.)},$$

which represents a straight line through

$$w = 0, \quad ax = cz,$$

i.e., the point of contact of the conic (1) in the plane $w = 0$ with $y = 0$. This line is repeated twice.

Again, the equation $\qquad C^2-4BD=0$

reduces to $\qquad \sqrt{-a\beta x}+\sqrt{(b\gamma-c\beta)\,z}+\sqrt{b\delta w}=0 \quad \text{...........(vii.)},$

which represents an escribed conic in the face $y=0$.

It is easily seen that the straight line (vi.) is a tangent to (vii.).

Similarly, the section by $z=0$ gives

$$C^2\,(B^2-4AC)=0,$$

or the line $\qquad -a\beta x+b\beta y-b\delta w=0 \quad \text{..................(viii.)},$

repeated twice, and the conic

$$\sqrt{a\gamma x}+\sqrt{(b\gamma-\beta c)\,y}+\sqrt{-c\delta w}=0 \quad \text{..............(ix.)}.$$

This is a conic escribed to the face $z=0$, and a tangent repeated twice.

The section by $w=0$ gives (v.), with the difference in (3) of making $w=0$. The result is the conic (1) and the line

$$-a\beta\gamma x+(b\gamma-c\beta)(\beta y-\gamma z)=0,$$

which is a tangent to the conic, repeated twice.

Similarly, the section of the developable surface by the plane $x=0$ is the line

$$(c\beta-b\gamma)(by-cz)-bc\delta w=0 \quad \text{....................(x.)},$$

repeated twice, and the conic (2).

It will be noticed that the section by each of the four faces of the fundamental tetrahedron is an inscribed (or escribed) conic, and a tangent to the conic repeated twice, the tangent cutting the three edges of the tetrahedron in the points of contact of the inscribed (or escribed) conics of the other three faces of the tetrahedron.

It will be observed that what has been proved of the tetrahedron of reference is true of a doubly infinite series of tetrahedrons.

If BC be the intersection of two faces of a tetrahedron in which conics are inscribed, we may state the same theorem for any tetrahedron $A'B'C'D'$, provided that B', C' are points in BC, and A', D' the points of intersection of tangents drawn to the two inscribed conics in their own planes from the points B' and C'.

It is interesting to observe the manner in which the developable surface degenerates when the inscribed conics touch each other.

Now the conics touch, if

$$\beta/b=\gamma/c=\lambda,\ \text{suppose};$$

in which case, if we write L for

$$-ax+by+cz-c\delta w/\gamma,$$

we have
$$A=\lambda c^2 y,\quad D=\lambda b^2 z,$$
$$B=\lambda\,(cL+bcy),$$
$$C=\lambda\,(bL+bcz).$$

It appears, therefore, that (v.) becomes a homogeneous equation in y, z, and L, and therefore represents a cone.

The equation becomes

$$(L^2-4bcyz)\,(L-by-cz)^2=0,$$

or the quadric $\quad (-ax+by+cz-c\delta w/\gamma)^2=4bcyz,$

which is a cone, and the plane

$$a\gamma x+c\delta w=0, \text{ repeated twice.}$$

For an account of some of Dr. Kleiber's results (p. 245), see a paper by him on "Some Differential Equations satisfied by the quantities K, E, &c., in Elliptic Functions" (*Messenger of Mathematics*, Vol. XVIII., pp. 167–184).

Prof. J. J. Thomson (see Vol. XIX., p. 591) has contributed a paper "On the Magnetic Effects produced by Motion in the Electric Field" (*Phil. Mag.* July, 1889, pp. 1–14).

With regard to our notice of Mr. John Brooksmith (Vol. XIX., p. 591), a correspondent, who was well acquainted with him, points out that, previous to entrance at Cambridge, Mr. Brooksmith passed the Session 1844–5 at University College, London, and that, before he went to Cheltenham, he had a year's experience in teaching at the Loretto School, near Edinburgh. In 1860 he published his "Arithmetic for Advanced Pupils, Part I." He mentions also that his friend was an Alderman and a great Freemason.

Mr. Ernest Temperley, Bursar and Assistant Tutor of Queens' College, Cambridge, was born on March 8th, 1849, and was educated at the Newcastle Grammar School till within two years of his going to Cambridge. During these two years he was first pupil, and then assistant master, at the Grammar School, Thornbury, Gloucestershire. In 1867 he gained a Minor Scholarship at Queens' College, where he commenced residence in October, 1867.

At the end of the first year he was promoted to a Foundation Scholarship, and, graduating in 1871, he was bracketed Fourth Wrangler, and obtained the second Smith's Prize. Mr. Temperley's

original intention was to go to the Bar, but, as he rapidly secured popularity as a private tutor, he changed his plans, married at an early age (1875), and settled down to a life in Cambridge.

In 1877 he was invited to fill the post of Lecturer in Mathematics and Natural Philosophy at Girton College. This post he continued to fill until the end of the Easter Term of 1887, when the state of his health compelled him to narrow his sphere of work. Among his pupils was Miss C. A. Scott, who attributed her success, in a large measure, to the instruction and guidance she received from him. In the years 1880 and 1881 he held the office of Senior Moderator, and in 1882 that of Examiner for the Mathematical Tripos. He contemplated writing a new treatise on the Differential and Integral Calculus, "as in the higher branches of the Integral he was not satisfied with any existing work." He has left some notes on this subject. He died January 8th, 1889.*

Mr. Temperley was elected a member of the Society, March 11th, 1880: he was never on the Council, nor did he contribute any papers to the Proceedings. On one or two occasions he gave valuable assistance as a referee.

Mr. William John Ibbetson was the youngest son of the Rev. Denzil John Holt Ibbetson, of St. John's, Adelaide, S. Australia; he was born January 6th, 1861, and died October 12th, 1879. He was educated at Haileybury College, where he held a Classical Scholarship, but was obliged to leave in 1875 on account of increasing deafness (he subsequently became totally deaf in 1877). He gained an Entrance Scholarship at Clare College, Cambridge, in 1879, and subsequently came out Seventeenth Wrangler in the first part of the Mathematical Tripos—"a place much below his merits," in the opinion of Mr. Mollison, his tutor. He then read for six months with Prof. J. J. Thomson, and was placed alone in the Second Division of the final part of the Mathematical Tripos. In 1886 he graduated M.A. He was a Fellow of the Royal Astronomical Society and the Cambridge Philosophical Society; he was also a member of the Institute of Actuaries and of the Cambridge Antiquarian and the Gipsy Lore Societies. Mr. Ibbetson was elected a member, December 11th, 1884. He contributed one paper to the Proceedings—"On the Airy-Maxwell Solution of the Equations of Equilibrium of an Isotropic Elastic Solid under Conservative Forces" (Vol. XVII.).

After taking his degree he devoted himself to the study of

* A full account of Mr. Temperley's career was furnished to the *Cambridge Review* for January 24th; it is to this sketch that, with the writer's permission, we are throughout indebted in the above notice.

Elasticity, and in 1887 published "An Elementary Treatise on the Mathematical Theory of Perfectly Elastic Solids; with a Short Account of Viscous Fluids" (pp. XIII., 515). This work is fully and favourably reviewed in *Nature* (Vol. XXXVII., pp. 97, 98). Mr. Ibbetson also contributed a letter to *Nature* (Vol. XXXII., pp. 76, 77) "On the Terminology of the Mathematical Theory of Elasticity." Mr. Mollison closes his letter, which we have freely quoted, with the remark,—"He was a man of great ability, and his book on Elasticity is most noteworthy, when it is remembered that he had been totally deaf from the age of sixteen, and acquired all his mathematics at such a terrible disadvantage." Mr. Perigal, who knew Mr. Ibbetson well, tells us, in addition to information quoted above, that Mr. Ibbetson communicated several papers to the above-named societies, and that he wrote also on the Romany language. He also states that Mr. Ibbetson was an Examiner in Mathematics at Haileybury College, and also in the Cambridge Local Examinations.

Professor Oscar Howard Mitchell was born in Locke, Ohio, October 4th, 1851, and died in Marietta, March 29th, 1889. He was the eldest of a family of five brothers and three sisters. His early life was spent on his father's farm, and his early education was acquired in the country schools. By private study, and the help of teaching, he was enabled to spend two years in the Mount Vernon High School, whence he entered the Freshman Class at Marietta in September, 1871.

After graduation, he was for three years Principal of the Marietta High School; he then entered on a course of advanced study in Mathematics and Logic at Johns Hopkins University, where he graduated in 1882 as Doctor of Philosophy. For three of his four years at the University, he held the honour of Fellow in Mathematics. He was elected a member of our Society, January 13th, 1881, and in the following year he was appointed to the Tyndall Fellowship, which would have enabled him to spend a year in scientific study abroad. He had, however, previously accepted a call to the Chair of Mathematics and Astronomy at Marietta, and so was obliged to decline the offer of the fellowship, an honour he highly prized. He appears to have been very successful as a teacher.*

Prof. Mitchell contributed the following papers to the *American Journal of Mathematics*:—

"On Binomial Congruences: comprising an extension of Fermat's and Wilson's Theorems, and a Theorem of which both are Special Cases," Vol. III., p. 294.

* The source of our information is contained in an article on Prof. Mitchell in the "Marietta College Olio," for April 20th, 1889.

"Some Theorems in Numbers," Vol. IV., p. 25.

"Note on Determinants of Powers," Vol. IV., p. 341.

He also furnished a paper on "A New Method of Symbolic Logic" to the "Studies in Logic, by members of the Johns Hopkins University."

Among the contributions to the "Brocardian" geometry which have appeared since our last notice (Vol. XVIII., p. 399), the most noteworthy are:—

"Premier Inventaire de la Géométrie du Triangle" (Association Française pour l'avancement des Sciences, Toulouse, 1887) and "Géométrie du Triangle" (Étude Bibliographique et Terminologique), (75 pp.,) an offprint from M. de Longchamps' *Journal de Math. Spéciales* (1887). These are by M. E. Vigarié.

M. Vigarié is a joint contributor with M. Lemoine to a "Note sur les Éléments Brocardiens," in the *Journal de Math. Élémentaires* (1888). M. Lemoine has written "Étude des points inverses" (Longchamps' *Journal,* 1887); "Questions diverses sur la nouvelle Géométrie du Triangle" (Ass. Fr., Nancy, 1886, and Toulouse, 1887); and "Quelques Questions se rapportant à l'étude des antiparallèles des côtés d'un triangle" (*Bulletin de la Société Math. de France,* 1886).

Dr. H. Lieber, of Stettin, in his annual school programme, has treated of this matter:—(1) "Über die Gegenmittellinie und den Grebe'schen Punkt" (1886); (2) "Über den Brocard'schen Kreis" (1887); (3) "Über den Tucker'schen Kreis," &c. (1888).

Prof. J. Neuberg, of Liège, has written:—"Sur les triangles équibrocardiens" (Ass. Fr., Oran, 1888).

Dr. Kiehl's programm (Realgymnasium zu Bromberg) is entitled, "Über die durch drei ähnliche Punktreihen erzeugten Dreiecke und Kegelschnitte" (1888).

Prof. Gob sends us an offprint from the *Mémoires de la Société Royale des Sciences de Liège,* in which he treats of the recent geometry under the titles, "Sur la Droite et le Cercle d'Euler," "Sur les Cercles de Neuberg" (1889).

It is sufficient here for us to mention that, in the *Journal d'Hoffmann* (Zeitschrift für Mathematischen, &c., 6[e] Fasc., 1889), Herr Schlömilch has written an article which has greatly excited French mathematicians. M. Vigarié has lent us a translation of it into French, "Crelle ou Brocard." The language is strong, and we do not wonder at the excitement it has created. M. Morel has replied to the attack in the *Journal de Math. Élémentaires,* and M. Vigarié also proposes to join in the contest.

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